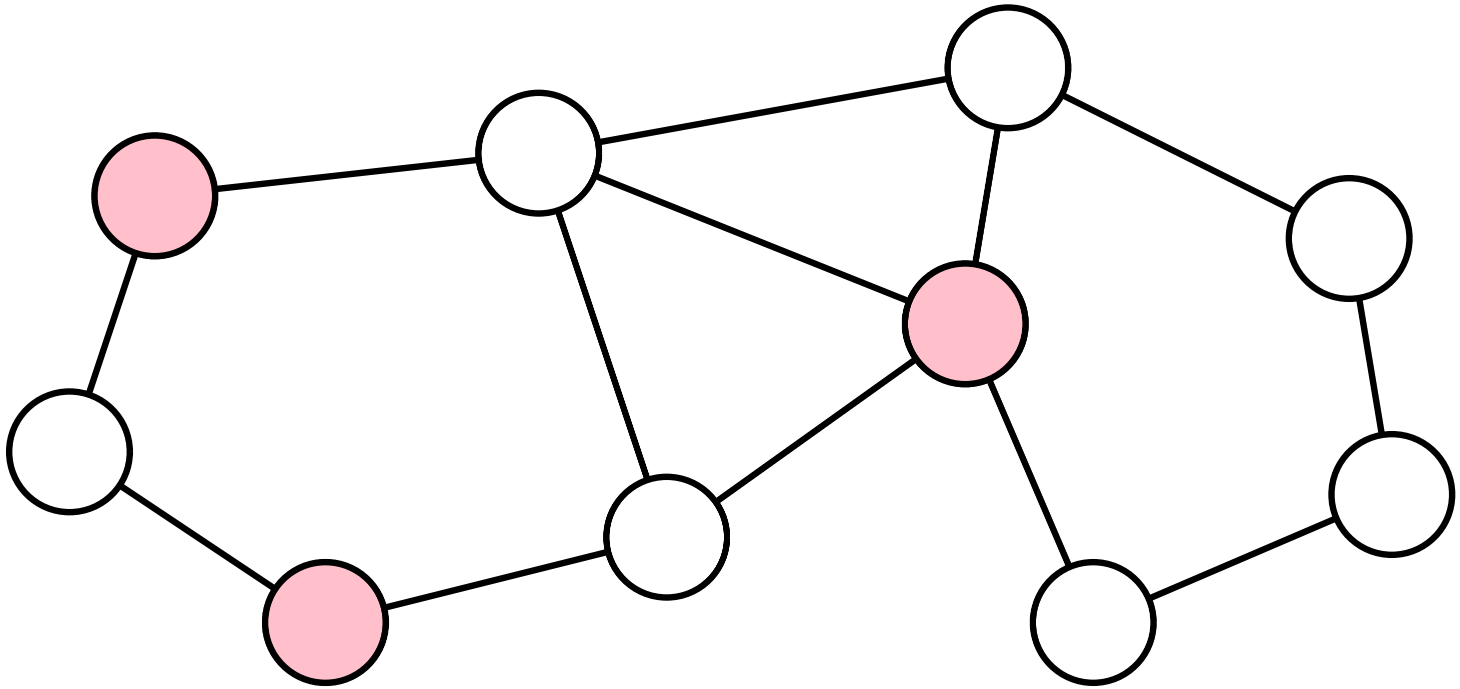
Luby's Maximal Independent Set Algorithm

Independent Sets



Definition: An *independent set* of a graph $G = (V, E)$ is a set $\mathcal{I} \subseteq V$ such that $\forall (u, v) \in E$, $u \notin \mathcal{I}$ or $v \notin \mathcal{I}$ (or both).

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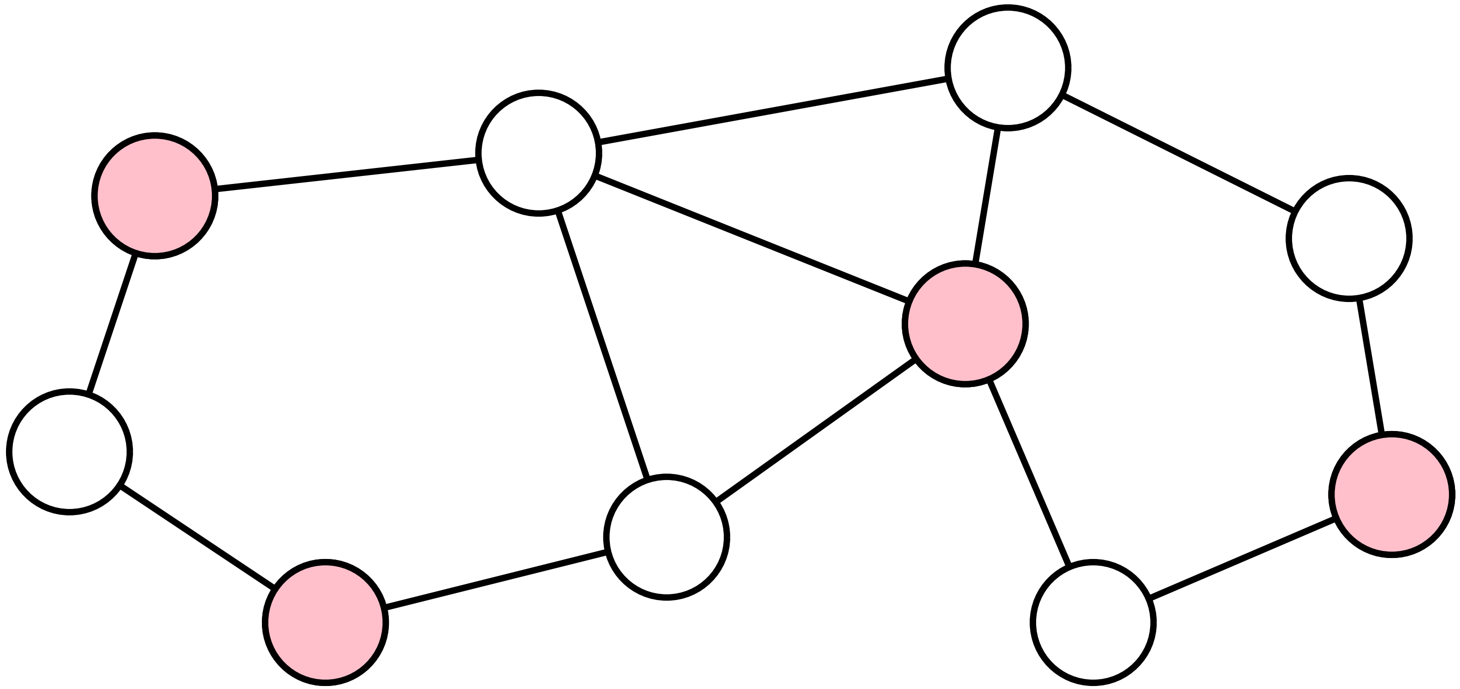

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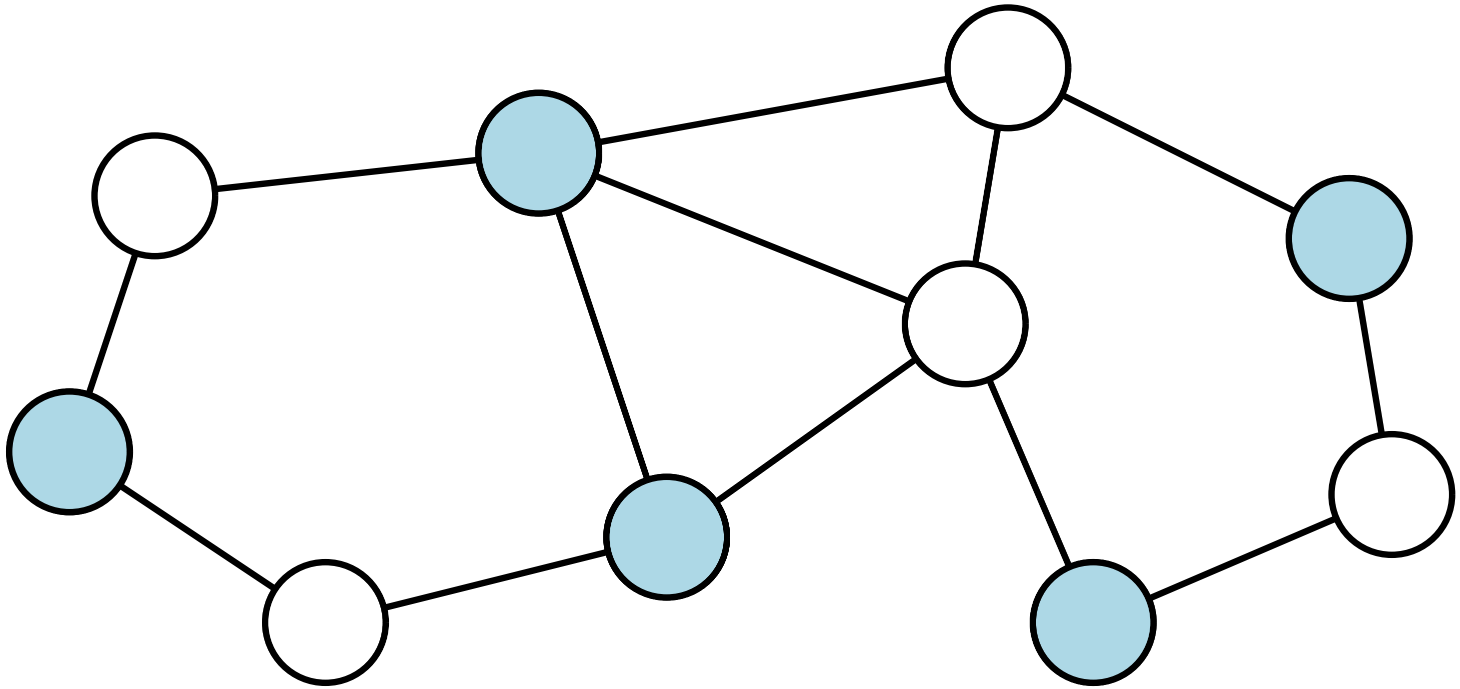

A **Maximal** Independent Set is *not necessarily* a **Maximum** Independent Set.

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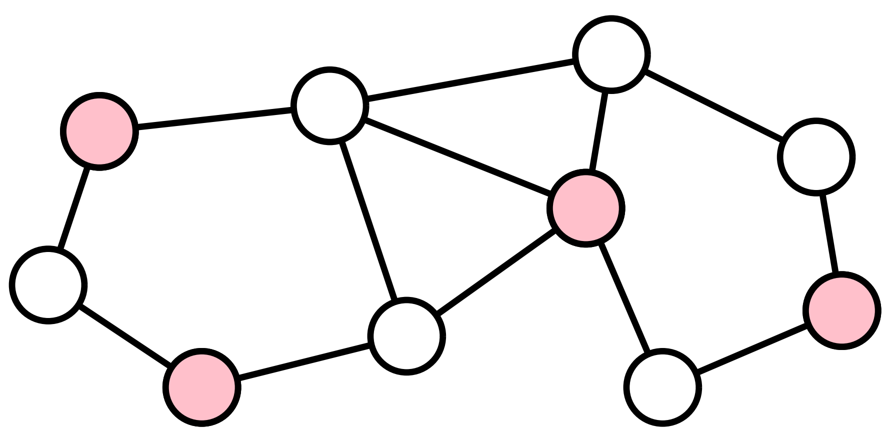
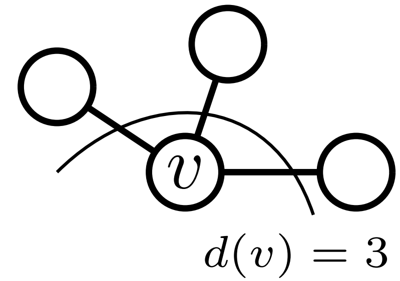
Luby's Algoritm

- A distributed algorithm to compute a Maximal Independent Set (MIS)
- Runs in time $O(\log d \cdot \log n)$ with high probability (w.h.p.), where d is the maximum degree of G.
- Asymptotically better than the algorithm of the previous lecture (which required $O(d \log n)$ time, w.h.p.).



Luby's Algoritm

Let $d(v)$ be the degree of vertex v in G.



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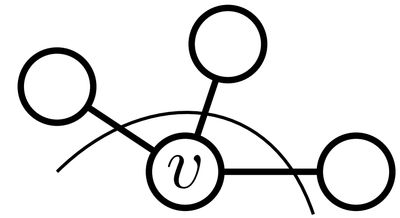

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At the generic phase k...

- Each node v in G_k *elects* itself with probability $p(v) = \frac{1}{2d(v)}$.

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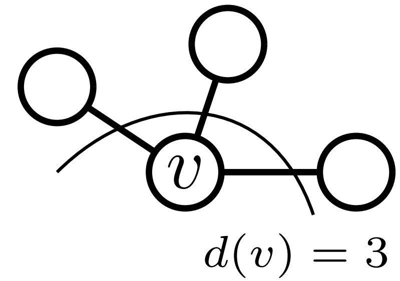

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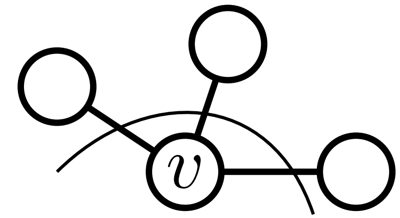

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- G_{k+1} is obtained by deleting the vertices in $\mathcal{I}_k$ and their neighbors $N(\mathcal{I}_k)$ from G_k

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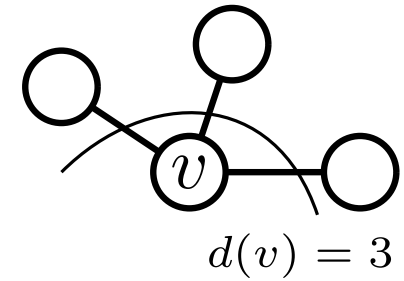

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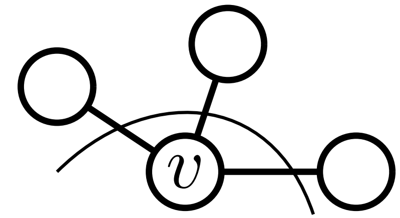

The algorithm works in *phases*

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At the generic phase k...

If v is a singleton, v always elects itself.

- Each node v in G_k *elects* itself with probability $p(v) = \frac{1}{2d(v)}$.
- Elected nodes are **candidates** to join an independent set $\mathcal{I}_k$ of G_k.
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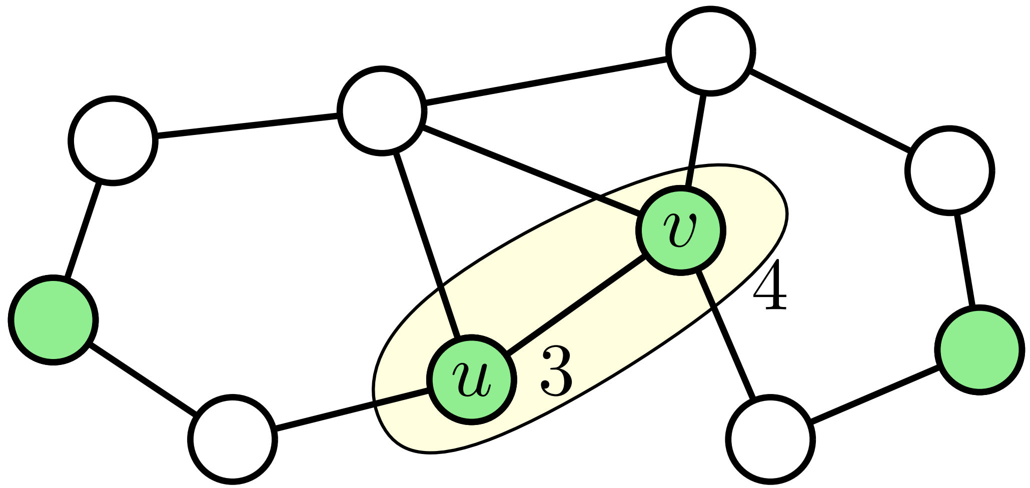

If two neighbors are elected simultaneously, the node with higher degree wins (remains in $\mathcal{I}_k$).

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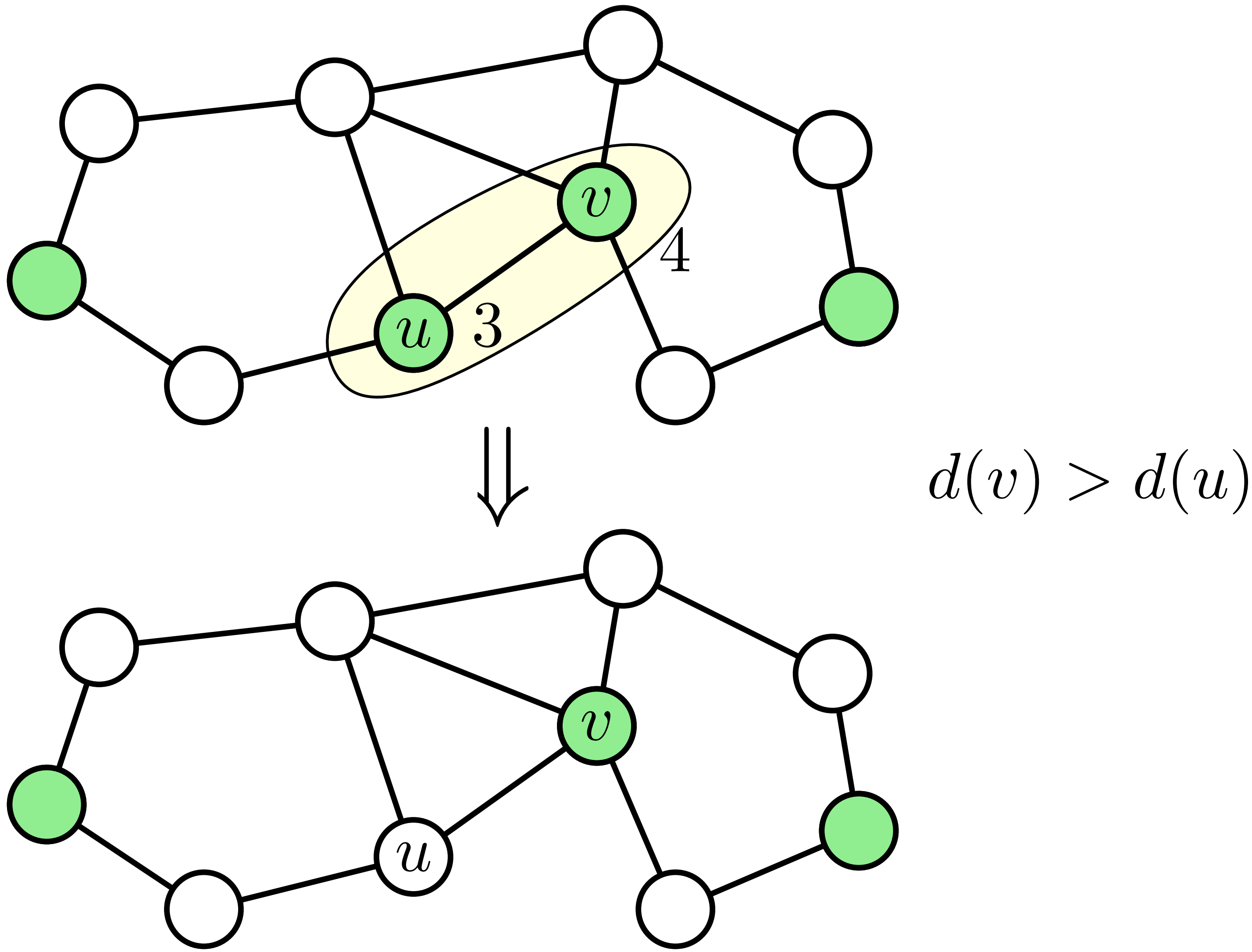

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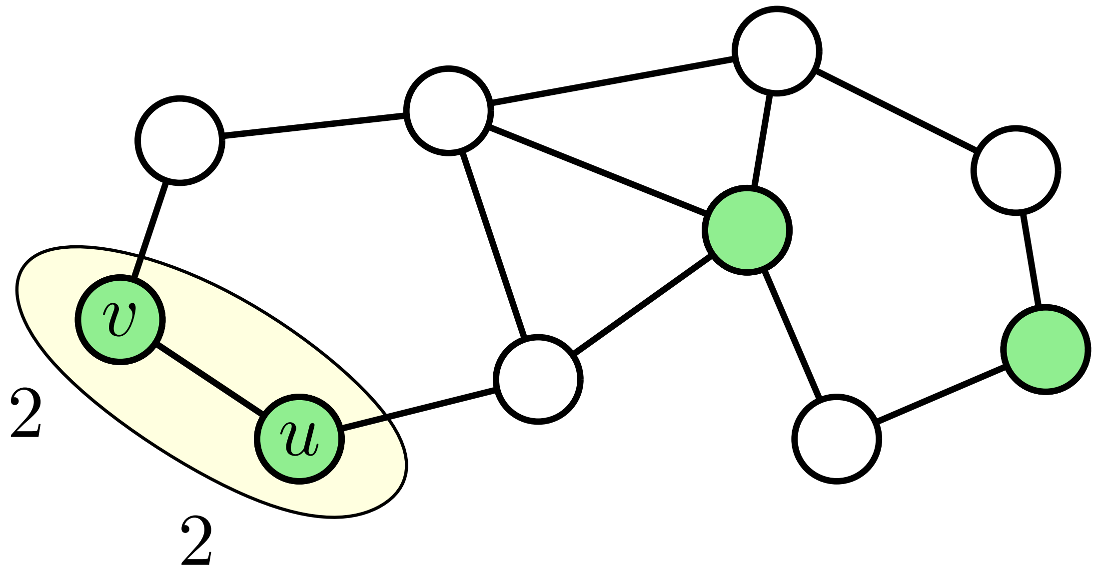
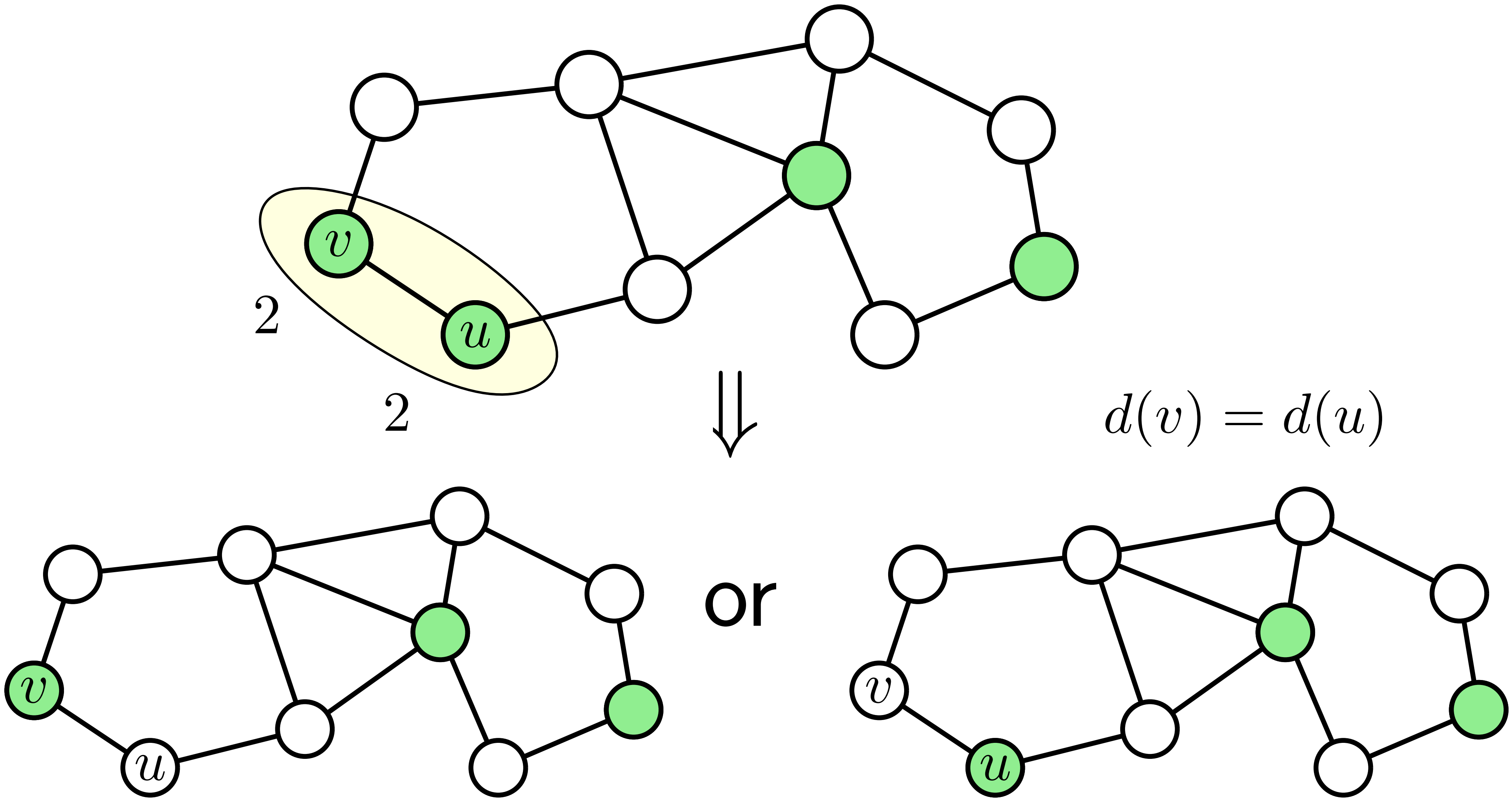


$d(v) = d(u)$

If both nodes have the same degree, break ties arbitrarily (e.g., by vertex ID).

Luby's Algoritm: Finding $\mathcal{I}_k$



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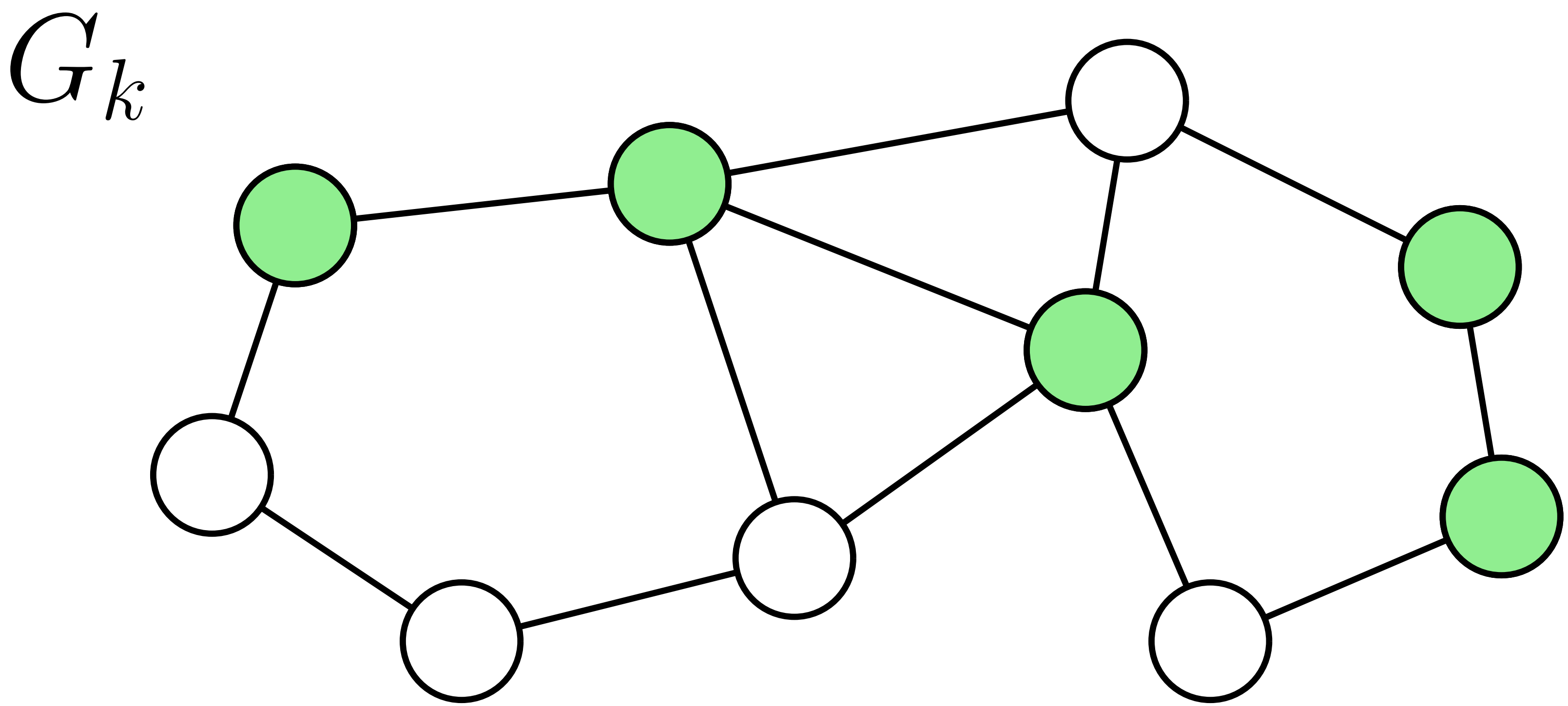
Luby's Algoritm

Previous rules are used to remove "problematic" nodes from the candidate nodes.



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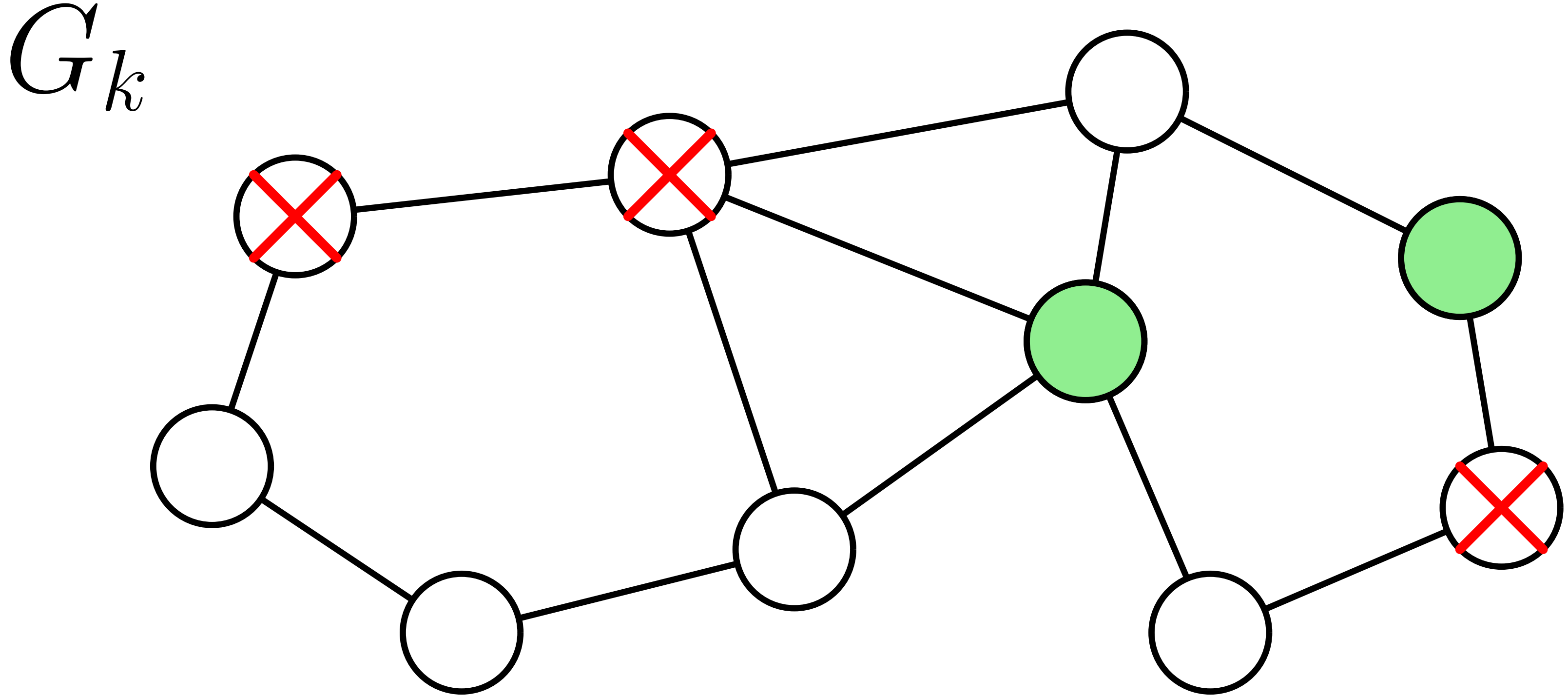

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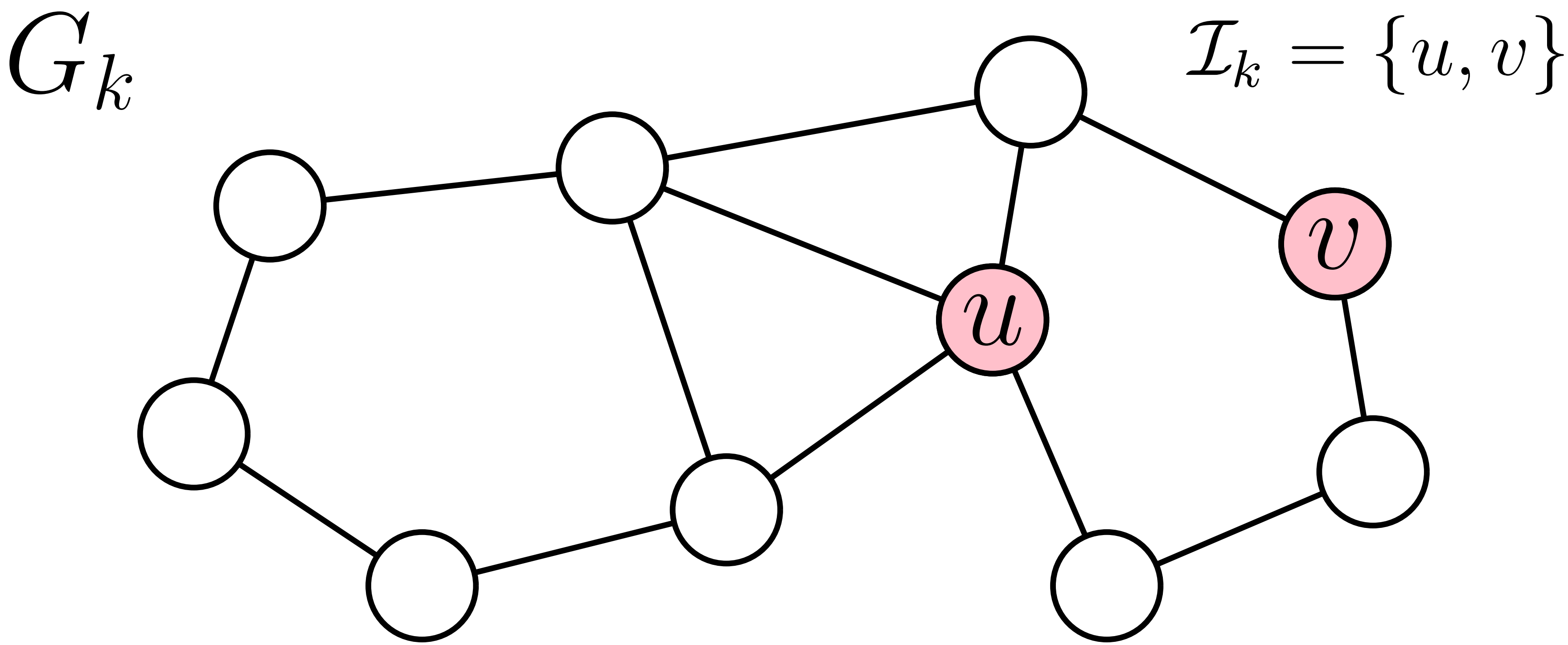

The remaning nodes form the independent set $\mathcal{I}_k$

Analysis

Consider the generic phase k

A *good event* H_v for node v is the following:

At least one neighbor of v *enters* $\mathcal{I}_k$ *(i.e.,* $\mathcal{I}_k \cap N(v) \neq \emptyset$*)*

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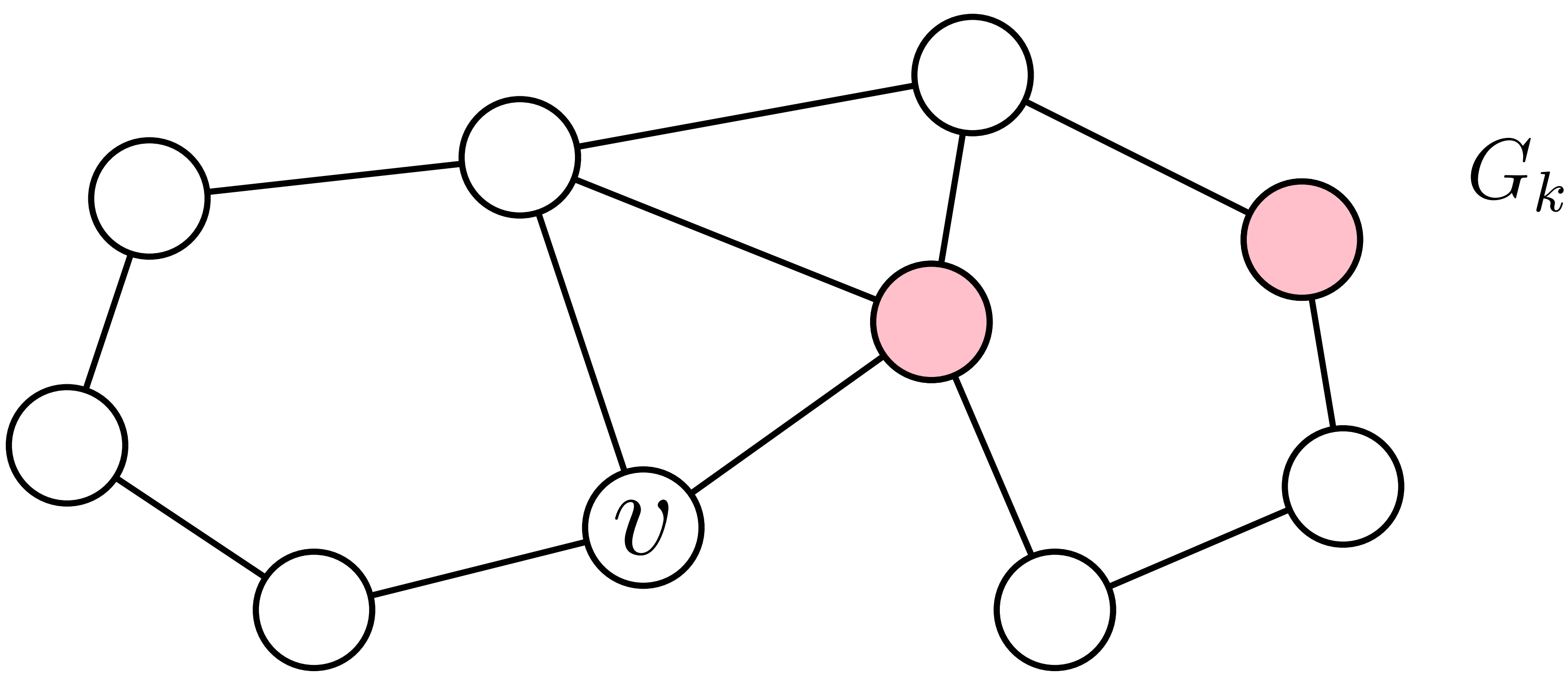
If H_v happens, then $v \in N(I_k)$ and v will not belong to G_{k+1}.



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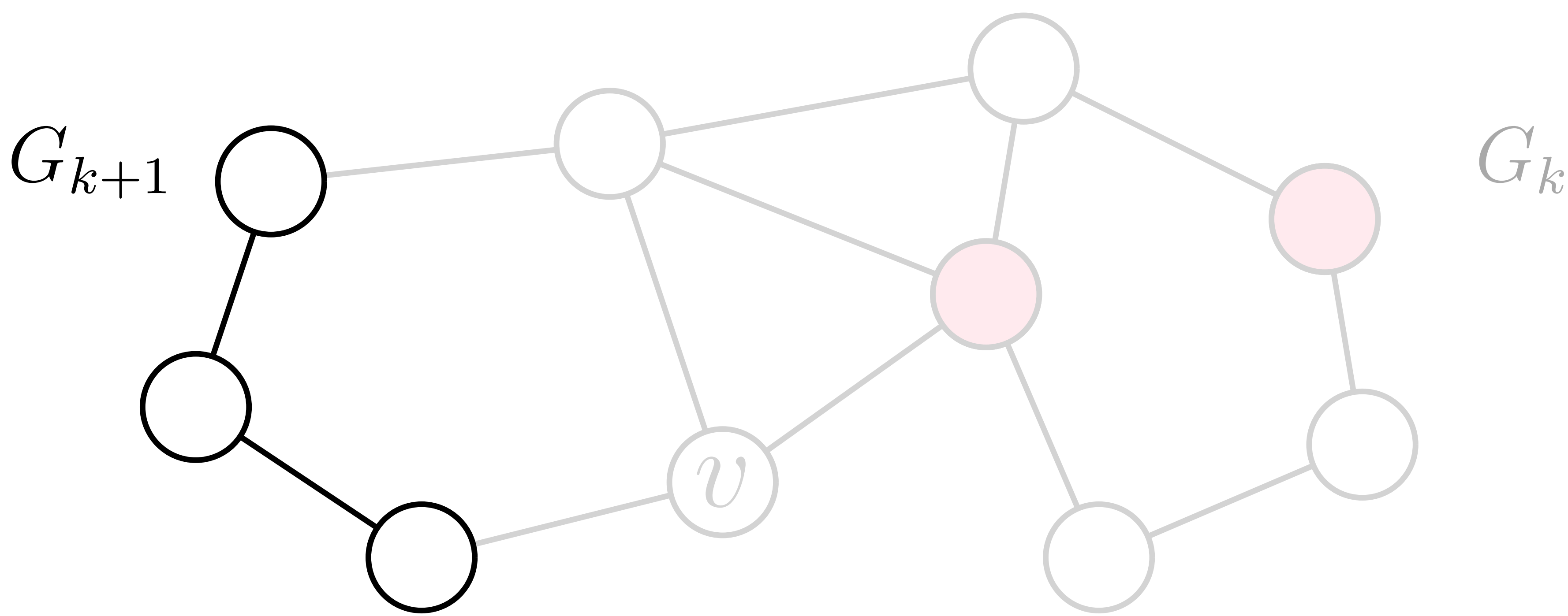

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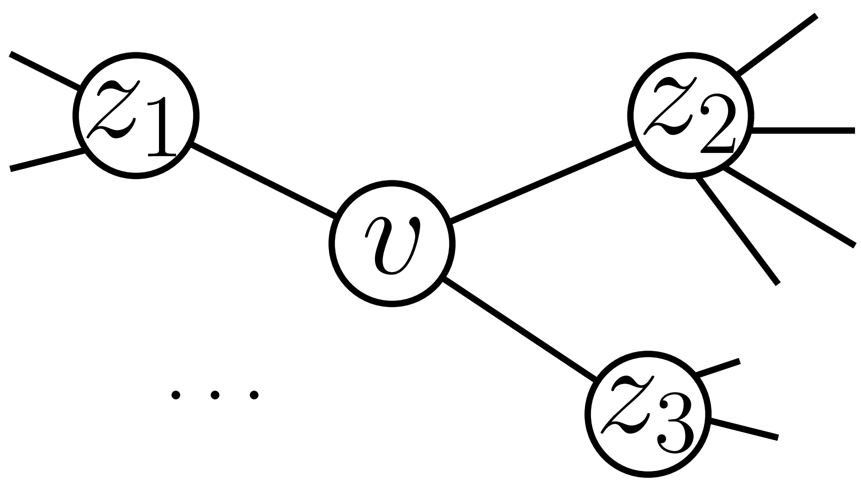
Lemma 1: At least one neighbor v elects itself with probability at least $1 - e^{-\frac{d(v)}{2\widetilde{d}(v)}}$,

where $\widetilde{d}(v) = \max_{z_i \in N(v)} d(z_i)$ is the maximum degree of a neighbor of v.



$d(v) = 3$

$\widetilde{d}(v) = 5$

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(Recall that elections are independent)

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$$\leq e^{-\frac{d(v)}{2\widetilde{d}(v)}}. \qquad \square$$

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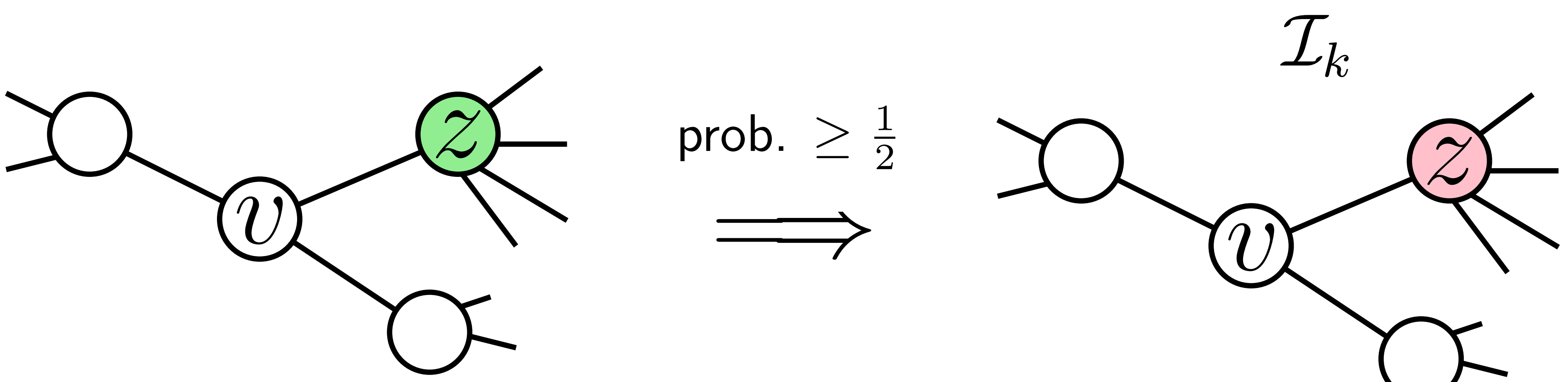
Lemma 2: If a node z elects itself, then it enters $\mathcal{I}_k$ with probability at least $\frac{1}{2}$.



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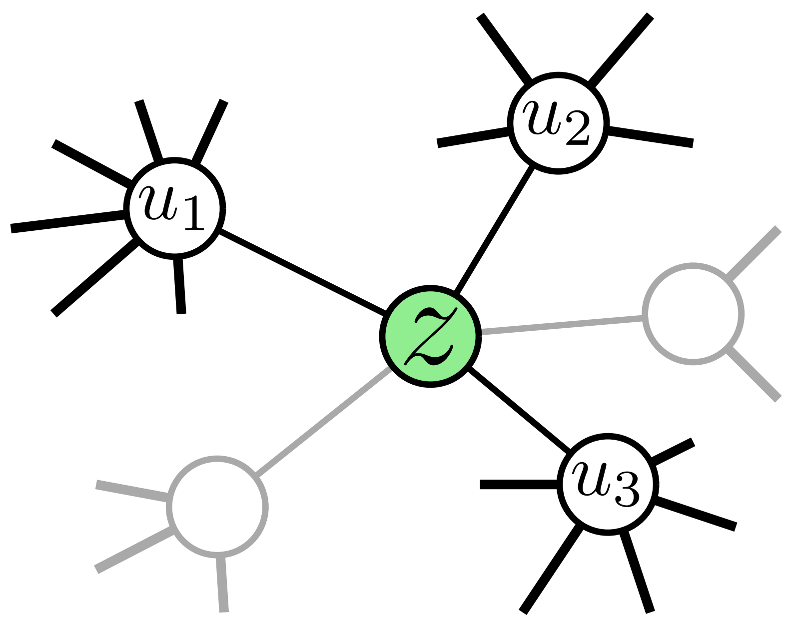

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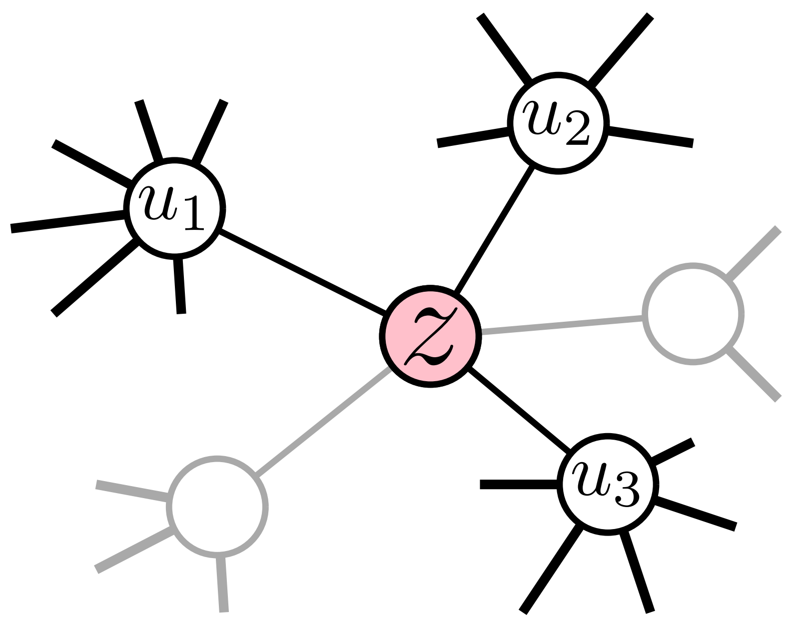

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Therefore, with probability at least $1 - \frac{1}{2} = \frac{1}{2}$, no u_i elects itself.

□

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Let $z_1, \ldots, z_{d(v)}$ be the neighbors of v.

Define $Y_i = \{z_i \in \mathcal{I}_k$ and no vertex $z_1, \ldots, z_{i-1}$ elects itself $\}$.

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$$P(H_v) \geq P\left(\bigcup_{i=1}^{d(v)} Y_i\right) = \sum_{i=1}^{d(v)} P(Y_i)$$

(since events $Y_1, \ldots, Y_{d(v)}$ are mutually exclusive)

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$Y_i = \{z_i \in \mathcal{I}_k$ and no vertex $z_1, \ldots, z_{i-1}$ elects itself $\}$.

Define:

$A_i = \{z_i$ elects itself and no vertex $z_1, \ldots, z_{i-1}$ elects itself $\}$.

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Therefore:

$$P(H_v) \geq \sum_{i=1}^{d(v)} P(Y_i) \geq \sum_{i=1}^{d(v)} P(A_i) P(z_i \in \mathcal{I}_k \mid B_i)$$

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A non-singleton vertex v in G_k "disappears" in phase k with (at least) the above probability.

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This is just some constant $c \approx 0.39$

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In a generic phase k, a node with degree at least $\frac{d_k}{2}$ disappears with probability at least c.

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$$(1-c)^\phi = (1-c)^{3 \log_{1-c} \frac{1}{n}} = (1-c)^{\log_{1-c} \frac{1}{n^3}} = \frac{1}{n^3}$$

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Thus, after $3 \log_{1-c} \frac{1}{n}$ phases, the probability that:

- v did not disappear; **and**
- the degree of v is still above $\frac{d_k}{2}$;

is at most $\frac{1}{n^3}$.

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The probabiltiy that after $3\log_{1-c}\frac{1}{n}$ phases there is **at least one node** with degree larger than $\frac{d_k}{2}$ is at most:

$$n \cdot \frac{1}{n^3} = \frac{1}{n^2}$$

Analysis

In other words:

Every $3 \log_{1-c} \frac{1}{n}$ phases the maximum degree of the graph **halves** with probability at least $1 - \frac{1}{n^2}$.

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$$\left(3\log_{1-c}\frac{1}{n}\right)\cdot\log d = O(\log d\cdot\log n)$$

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Every $3\log_{1-c}\frac{1}{n}$ phases the maximum degree of the graph **halves** with probability at least $1-\frac{1}{n^2}$.

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MIS forms in $O(\log d \cdot \log n)$ phases with probability at least $1-\frac{1}{n}$.

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- Probability of success: $\geq 1 - \frac{1}{n}$