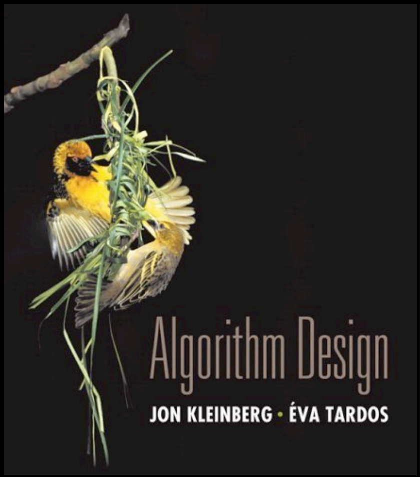


Chapter 13

Randomized Algorithms

Randomization

Algorithmic design patterns.
- Greedy.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- Randomization.

in practice, access to a pseudo-random number generator

Randomization. Allow fair coin flip in unit time.

Why randomize? Can lead to simplest, fastest, or only known algorithm for a particular problem.

Ex. Symmetry breaking protocols, graph algorithms, quicksort, hashing, load balancing, Monte Carlo integration, cryptography.

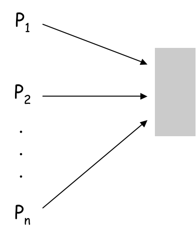
13.1 Contention Resolution

Contention Resolution in a Distributed System

Contention resolution. Given n processes $P_1, \ldots, P_n$, each competing for access to a shared database. If two or more processes access the database simultaneously, all processes are locked out. Devise protocol to ensure all processes get through on a regular basis.

Restriction. Processes can't communicate.

Challenge. Need symmetry-breaking paradigm.



Protocol. Each process requests access to the database at time t with probability p = 1/n.

Claim. Let S[i, t] = event that process i succeeds in accessing the database at time t. Then $1/(e \cdot n) \le \Pr[S(i, t)] \le 1/(2n)$.

Pf. By independence, $\Pr[S(i, t)] = p\,(1-p)^{n-1}$.

process i requests access (p); none of remaining n-1 processes request access ($(1-p)^{n-1}$)

- Setting p = 1/n, we have $\Pr[S(i, t)] = 1/n\,(1 - 1/n)^{n-1}$. ▪

value that maximizes Pr[S(i, t)]; between 1/e and 1/2

Useful facts from calculus. As n increases from 2, the function:
- $(1 - 1/n)^n$ converges monotonically from 1/4 up to 1/e
- $(1 - 1/n)^{n-1}$ converges monotonically from 1/2 down to 1/e.

Claim. The probability that process i fails to access the database in en rounds is at most 1/e. After $e \cdot n(c \ln n)$ rounds, the probability is at most n^{-c}.

Pf. Let F[i, t] = event that process i fails to access database in rounds 1 through t. By independence and previous claim, we have $\Pr[F(i, t)] \le (1 - 1/(en))^t$.

- Choose $t = \lceil e \cdot n \rceil$: $\Pr[F(i,t)] \le \left(1-\frac{1}{en}\right)^{\lceil en \rceil} \le \left(1-\frac{1}{en}\right)^{en} \le \frac{1}{e}$
- Choose $t = \lceil e \cdot n \rceil \lceil c \ln n \rceil$: $\Pr[F(i,t)] \le \left(\frac{1}{e}\right)^{c \ln n} = n^{-c}$

Claim. The probability that all processes succeed within $2e \cdot n \ln n$ rounds is at least 1 - 1/n.

Pf. Let F[t] = event that at least one of the n processes fails to access database in any of the rounds 1 through t.

$$\Pr[F[t]] = \Pr\left[\bigcup_{i=1}^{n} F[i,t]\right] \le \sum_{i=1}^{n} \Pr[F[i,t]] \le n\left(1-\frac{1}{en}\right)^t$$

union bound; previous slide

- Choosing $t = 2\lceil en \rceil \lceil c \ln n \rceil$ yields $\Pr[F[t]] \le n \cdot n^{-2} = 1/n$. ▪

Union bound. Given events $E_1, \ldots, E_n$, $\Pr\left[\bigcup_{i=1}^{n} E_i\right] \le \sum_{i=1}^{n} \Pr[E_i]$

13.2 Global Minimum Cut