Algorithms for Concurrent Distributed Systems:

The Mutual Exclusion problem

Concurrent Distributed Systems

Changes to the model from the MPS:

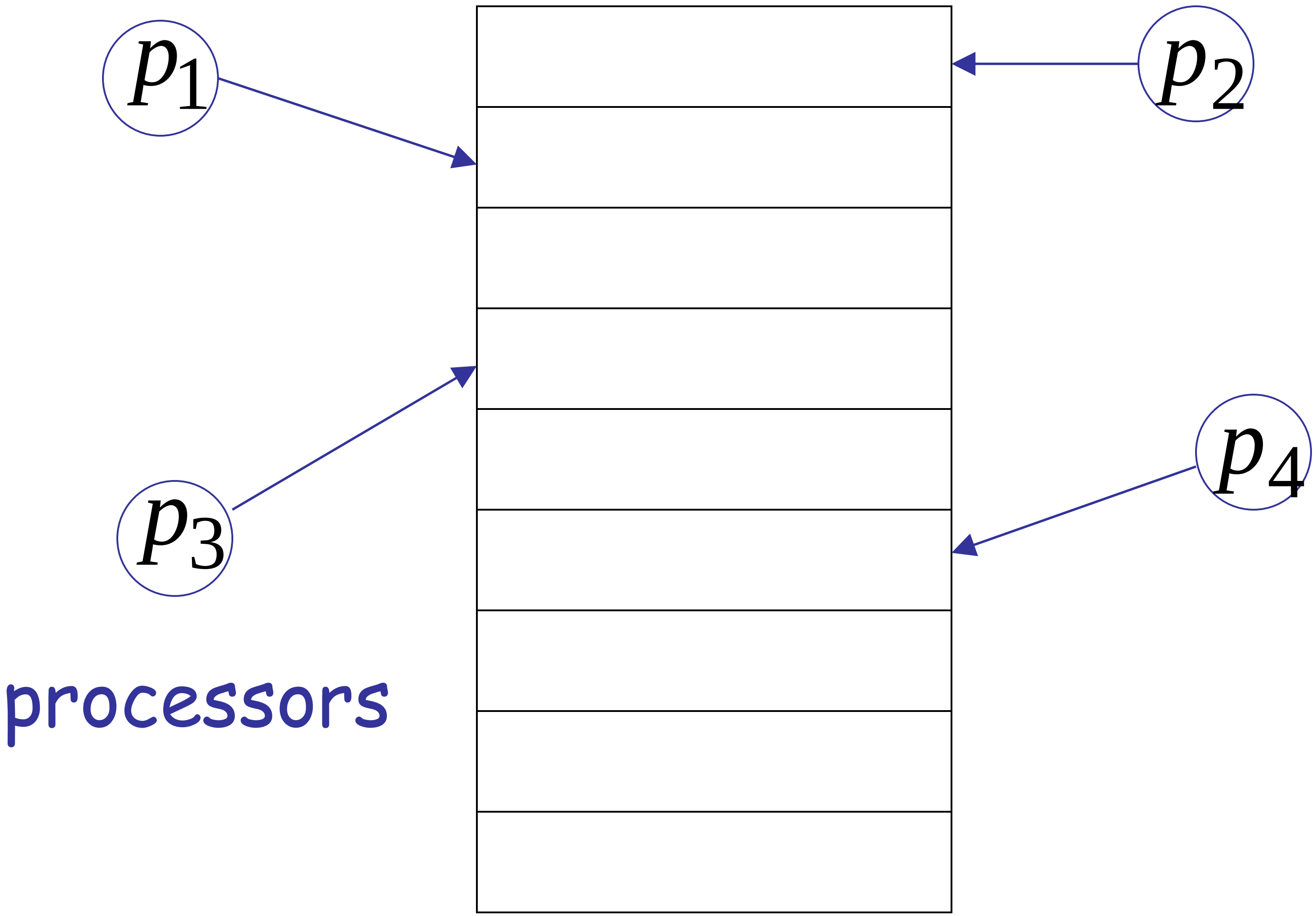
- MPS basically models a distributed system in which processors needs to coordinate to perform a widesystem goal (e.g., electing their leader)
- Now, we turn our attention to concurrent systems, where the processors run in parallel but without necessarily cooperating (for instance, they might just be a set of laptops in a LAN)

Shared Memory System (SMS)

Changes to the model from the MPS:

- No cooperation $\Rightarrow$ no **communication** channels between processors and no *inbuf* and *outbuf* state components
- Processors **notificate** their status via a set of shared variables, instead of passing messages $\Rightarrow$ no any communication graph!
- Each shared variable has a **type**, defining a set of operations that can be performed **atomically** (i.e., **instantaneously**, without interferences)

Shared Memory



Types of Shared Variables

1. Read/Write
2. Read-Modify-Write
3. Test & Set
4. Compare-and-swap

   .
   .
   .

We will focus on the **Read/Write** type (the simplest one to be realized)

Read/Write Variables

Read(v)	Write(v,a)
return(v);	v := a;

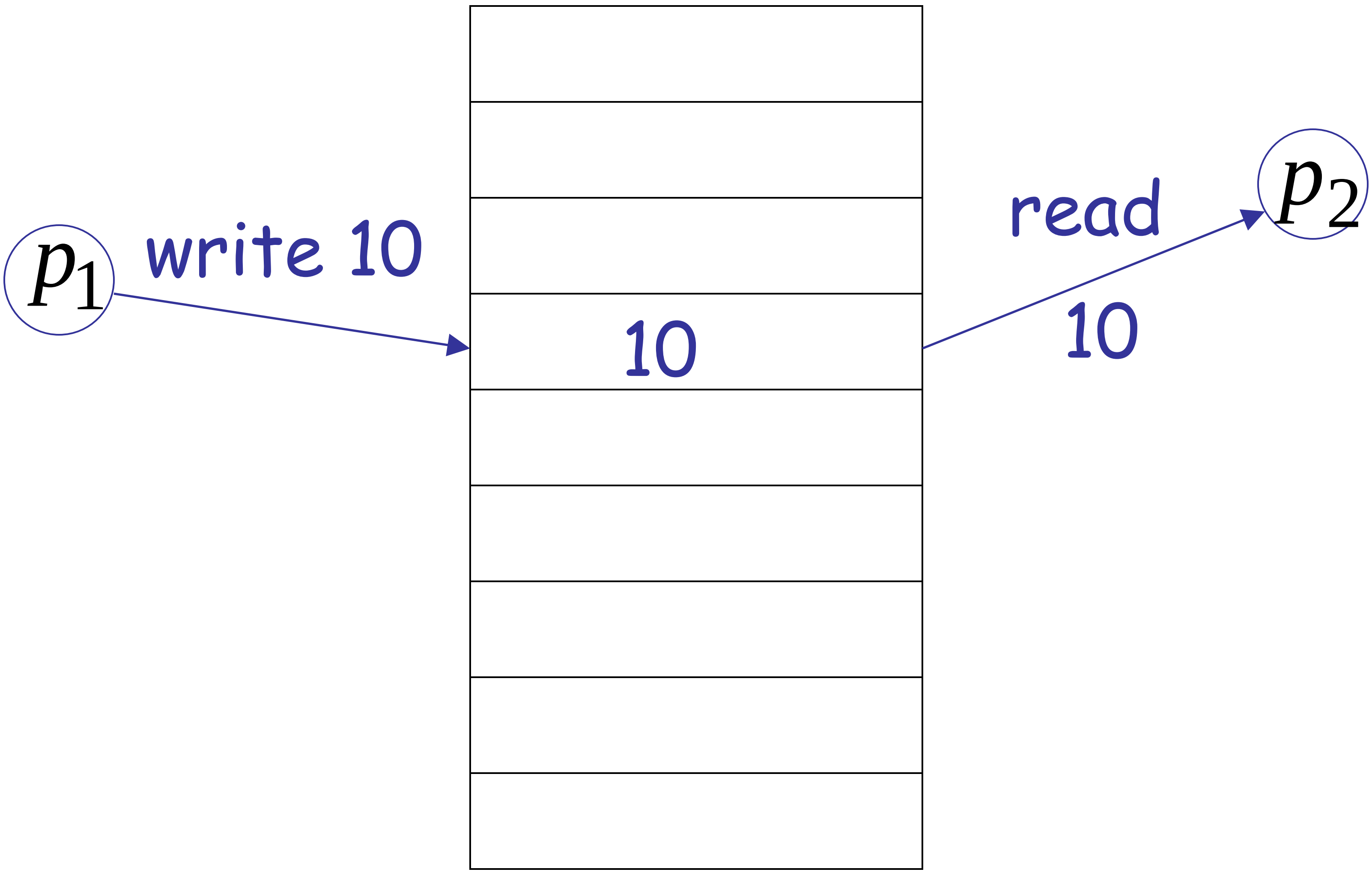
In one atomic step a processor can:

- read the variable, or
- write the variable

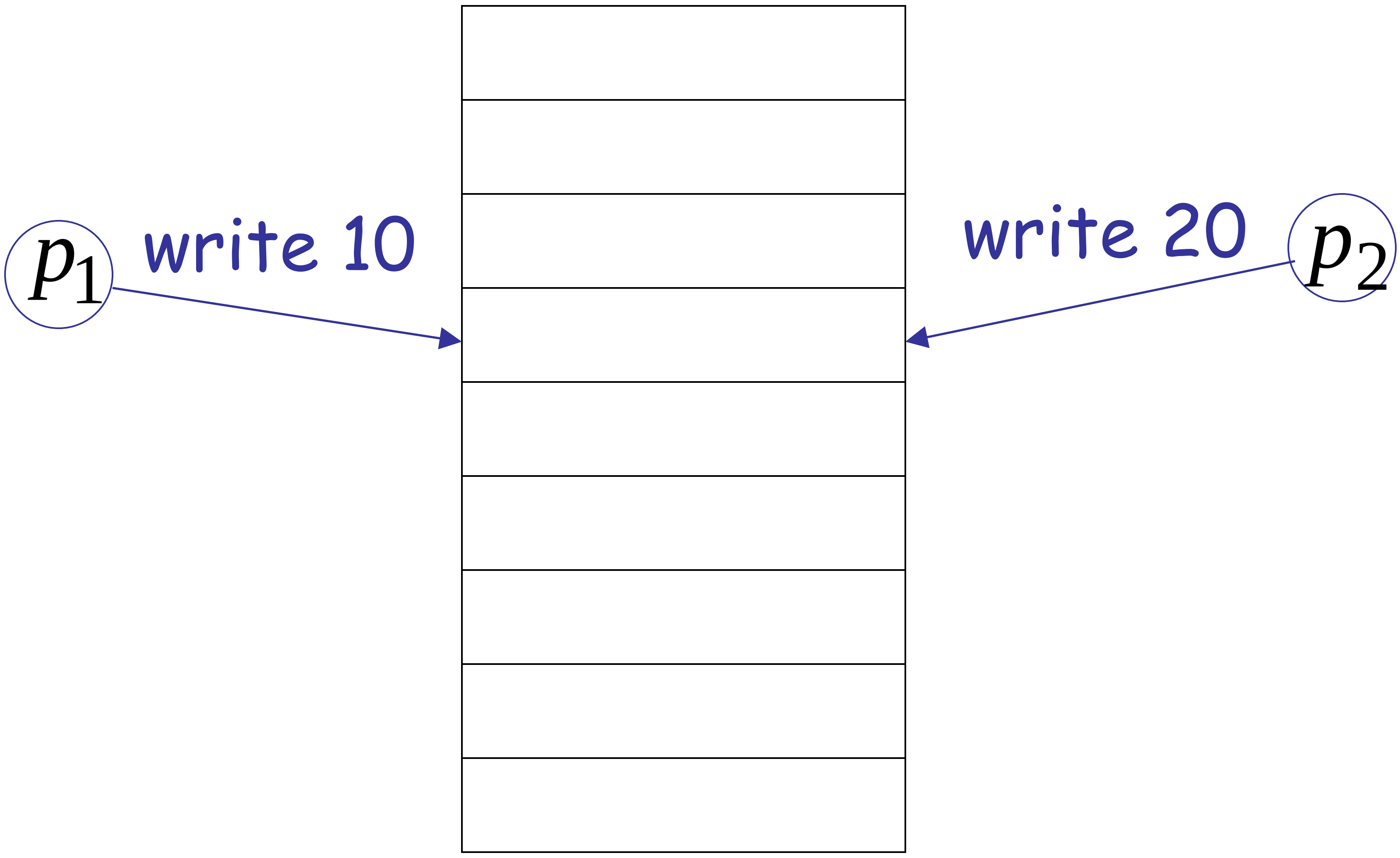
... but not both!

⇒ between a read and a write operation by a given processor, some other processor could make other operations on the variable!

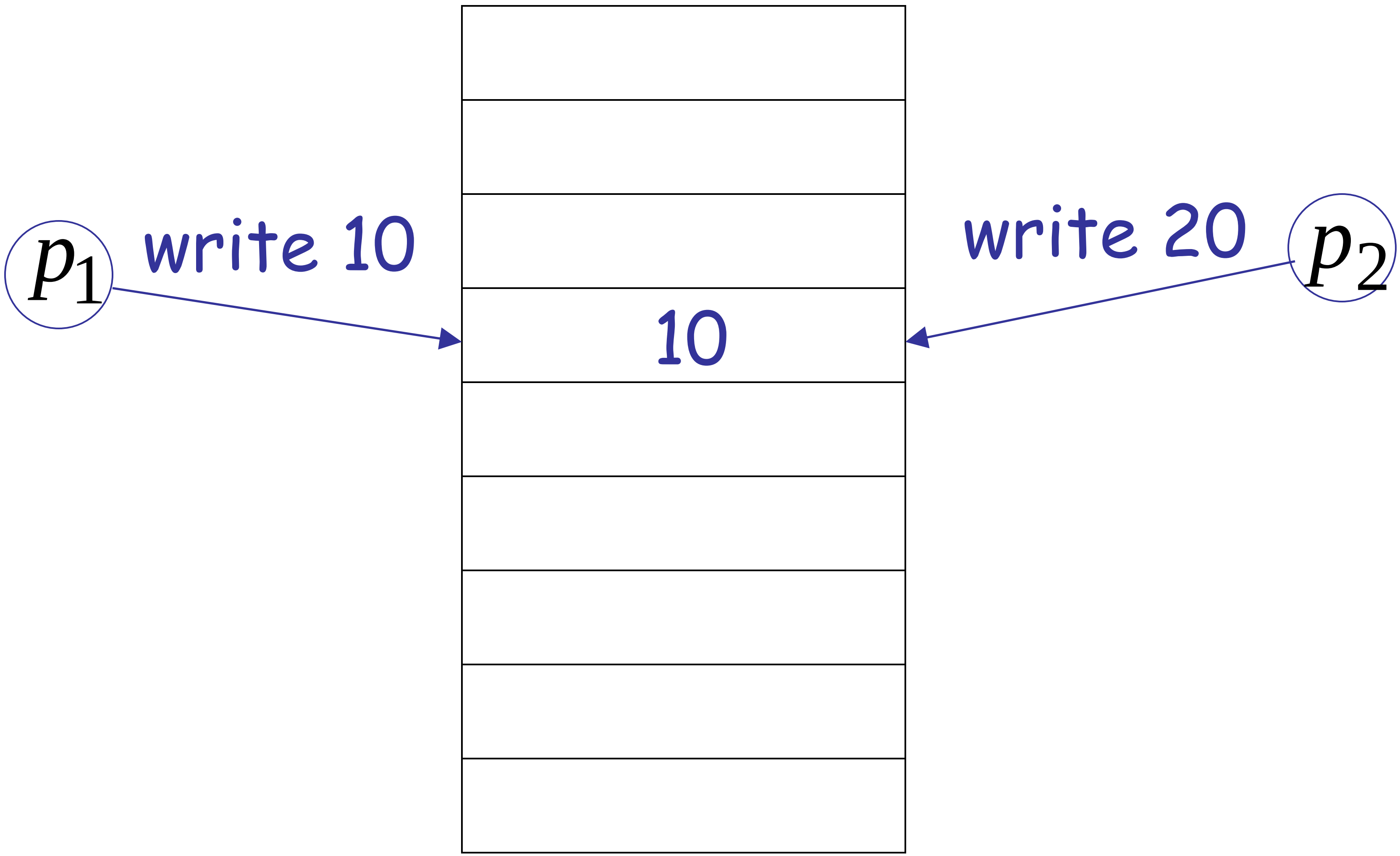
An example



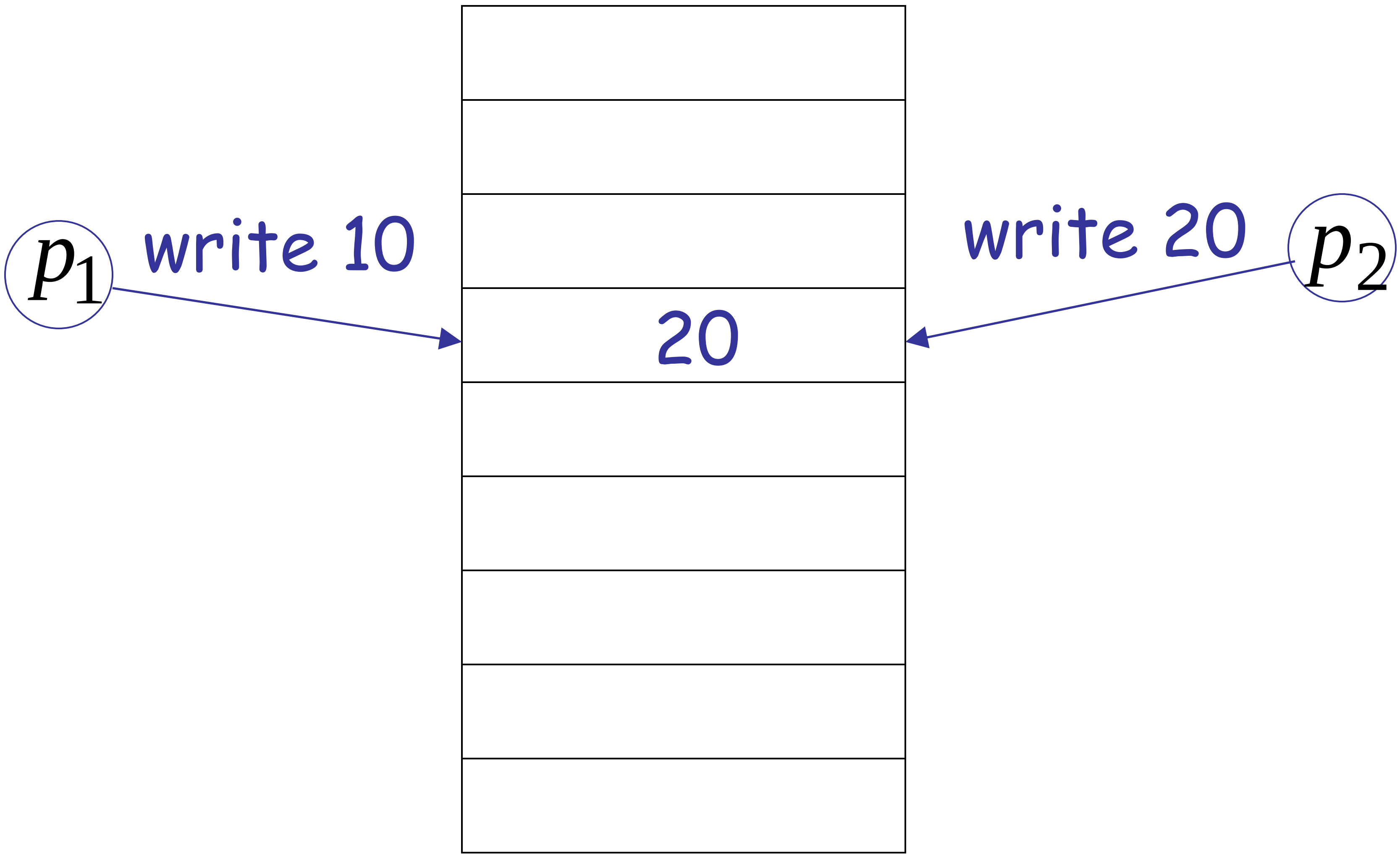
Simultaneous writes



Simultaneous writes are scheduled: Possibility 1

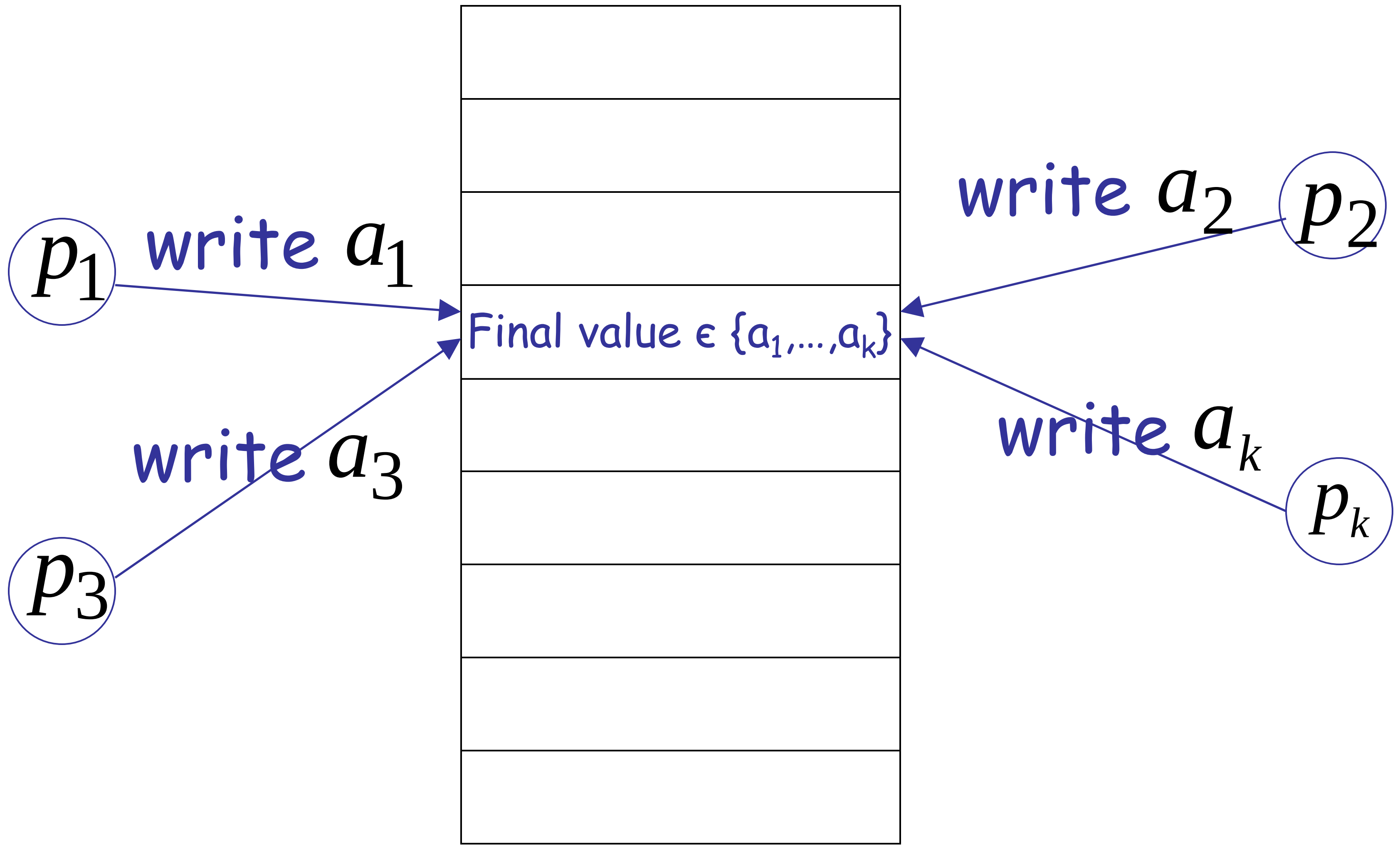


Simultaneous writes are scheduled: Possibility 2

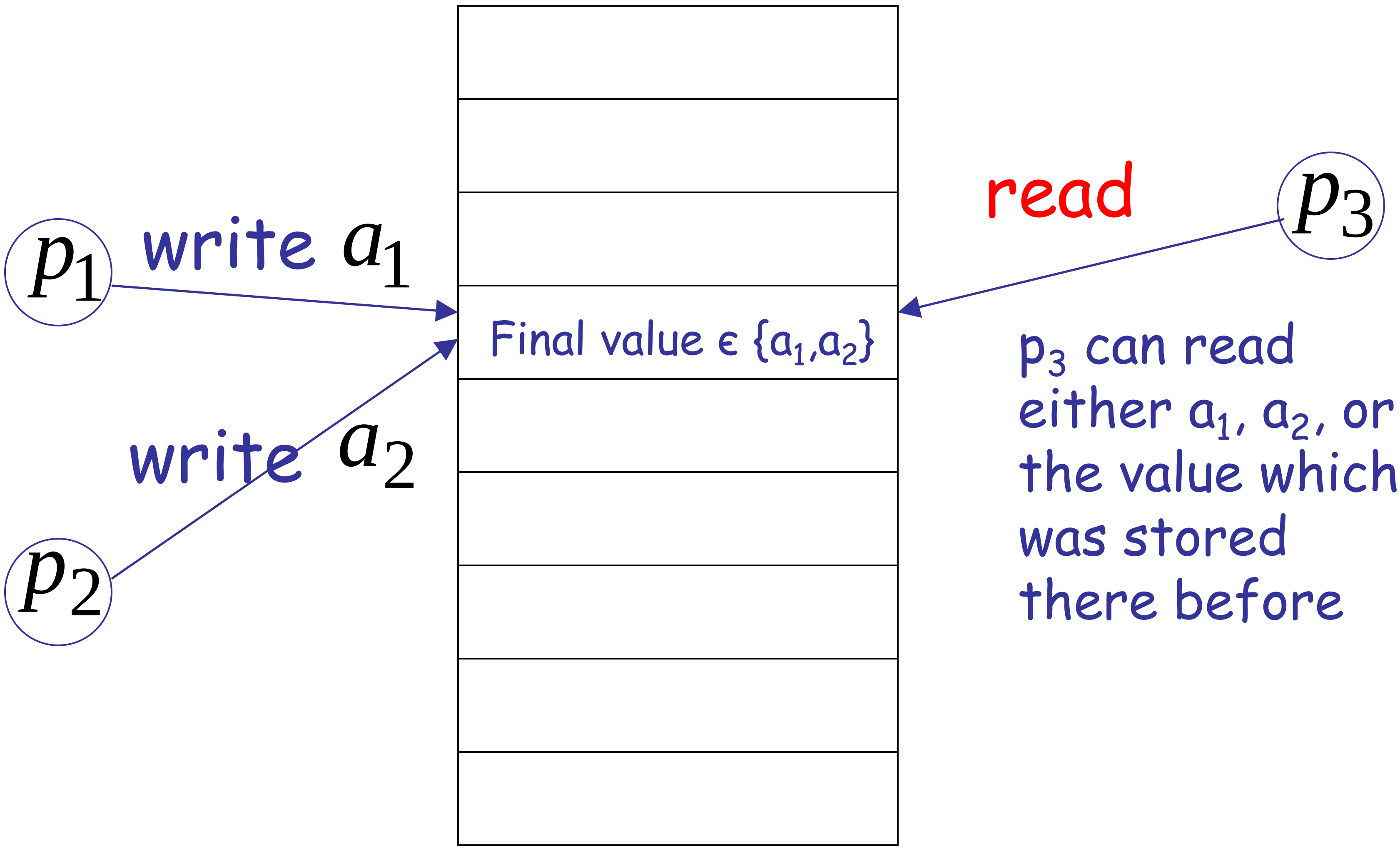


$\Rightarrow$ the surviving value is arbitrary!

Simultaneous writes are scheduled: In general:

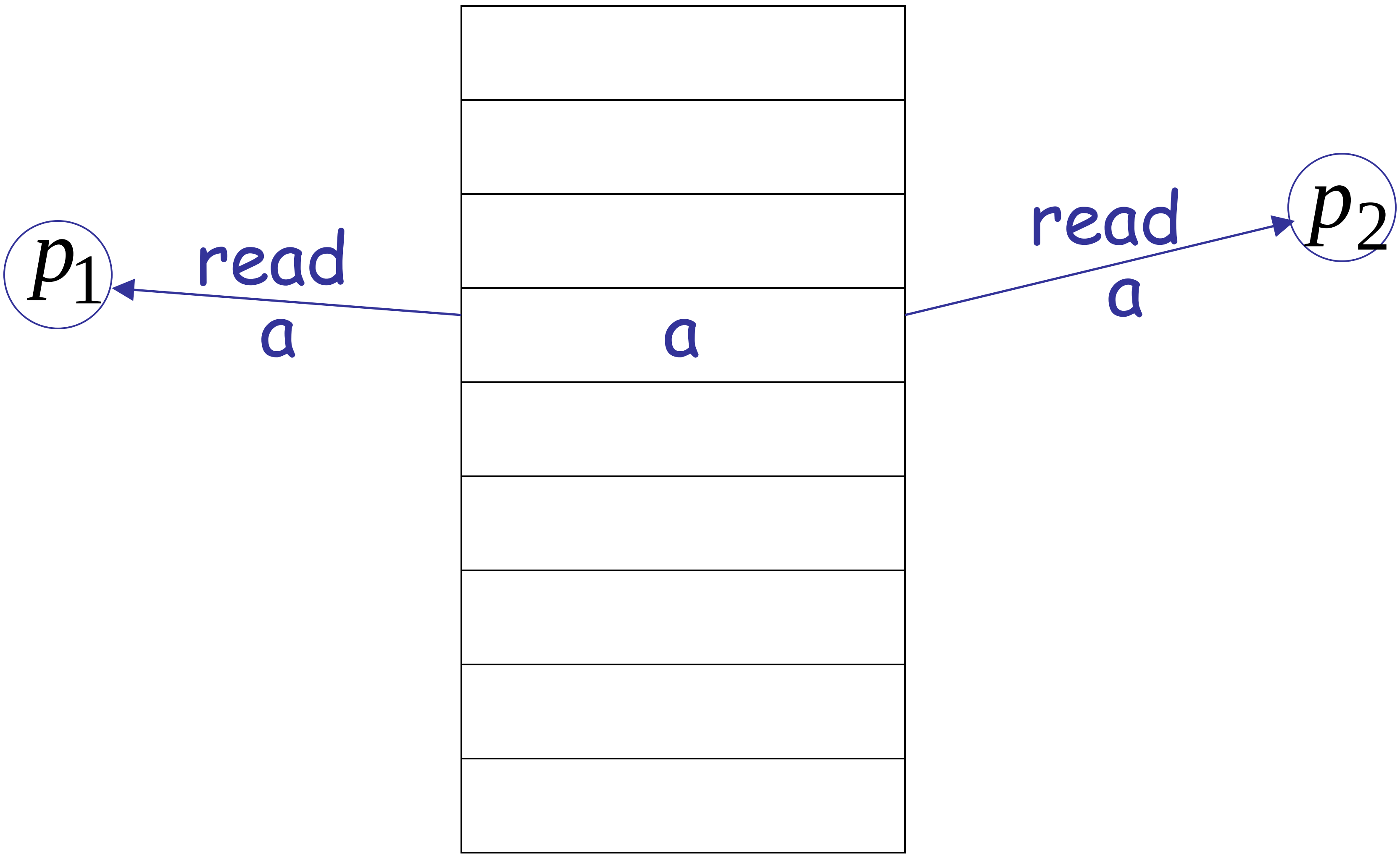


Simultaneous reads and writes are also scheduled



p_3 can read either a_1, a_2, or the value which was stored there before

Simultaneous Reads: no problem!



All read the same value

Computation Step in the Shared Memory Model

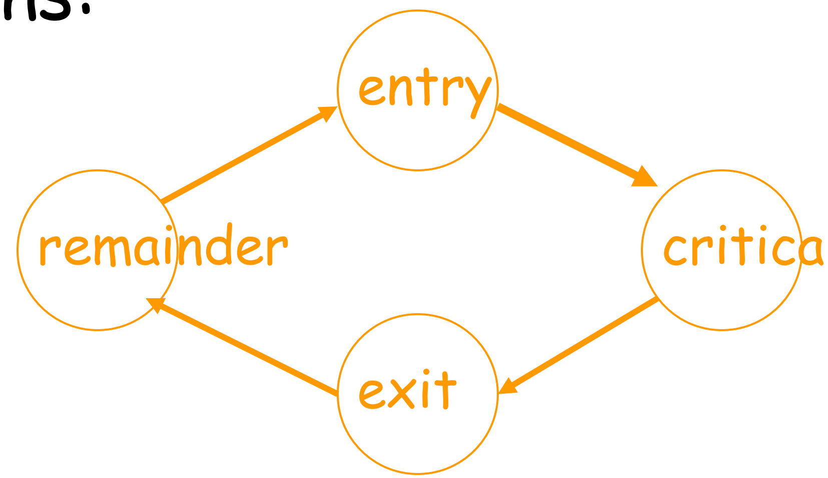
- When processor p_i takes a step:
  - p_i 's state in the old configuration specifies which shared variable is to be accessed and with which operation
  - operation is done: shared variable's value in the new configuration (possibly) changes according to the operation
  - p_i 's state in the new configuration changes according to its old state and the result of the operation

The mutual exclusion (mutex) problem

- The main challenge in managing concurrent systems is coordinating access to resources that are shared among processes
- Assumptions on the SMS (similarly to the MPS):
  - Non-anonymous (ids are in [0..n-1])
  - Non-uniform
  - Asynchronous

Mutex code sections

- Each processor's code is divided into four sections:



  - **entry:** synchronize with others to ensure mutually exclusive access to the …
  - **critical:** use some resource; when done, enter the…
  - **exit:** clean up; when done, enter the…
  - **remainder:** not interested in using the resource

Mutex Algorithms

- *A mutual exclusion algorithm* specifies code for entry and exit sections to ensure:
  - mutual exclusion: at most one processor is within the critical section at any time, and
  - some kind of liveness condition, i.e., a guarantee on the use of the critical section (under the general assumption that no processor stays in its critical section forever). There are three commonly considered ones:

Mutex Liveness Conditions

- **no deadlock:** if a processor is in its entry section at some time, then later *some* (i.e., maybe another) processor is in its critical section (notice that a processor can be starved/locked in this situation)
- **no lockout:** if a processor is in its entry section at some time, then later the *same* processor is in its critical section (but maybe it will be surpassed an unbounded number of times by other processors)
- **bounded waiting:** no lockout + while a processor is in its entry section, any other processor can enter the critical section no more than a bounded number of times.

These conditions are increasingly strong: bounded waiting $\Rightarrow$ no lockout $\Rightarrow$ no deadlock

Complexity Measure for Mutex

- Main complexity measure of interest for shared memory mutex algorithms is amount of shared space needed.
- Space complexity is affected by:
  - how powerful is the type of the shared variables (recall we only focus on Read/Write type)
  - how strong is the liveness condition to be satisfied (no deadlock/no lockout/bounded waiting)

Mutex Results Using R/W

Liveness Condition	***upper bound***	***lower bound***
no deadlock		n booleans
no lockout	$3(n-1)$ booleans (for $n=2^k$) (tournament algorithm)	
bounded waiting	n booleans + n <u>unbounded</u> integers (bakery algorithm)	

The Bakery Algorithm (L. Lamport, 1974)

- Guaranteeing:
  - Mutual exclusion
  - Bounded waiting
- Using 2n shared read/write variables
  - booleans `Choosing[i]`: initially false, written by p_i and read by others
  - (unbounded) integers `Number[i]`: initially 0, written by p_i and read by others

Bakery Algorithm

Code for entry section of p_i:

```
Choosing[i] := true
Number[i] := max{Number[0],...,
                      Number[n-1]} + 1
Choosing[i] := false
for j := 0 to n-1 (except i) do
    wait until Choosing[j] = false
    wait until Number[j] = 0 or
     (Number[j],j) > (Number[i],i)
endfor
```

Doorway subsection (DS)

Bakery subsection (BS)

Semaphores

Ticket of p_i

Code for exit section of p_i:

```
Number[i] := 0
```

BA Provides Mutual Exclusion

Lemma 1: If p_i is in the critical section (CS), then `Number`[i] > 0.

Proof: Trivial.

Lemma 2: If p_i is in the CS and `Number`[k] ≠ 0 (k ≠ i), then (`Number`[k],k) > (`Number`[i],i).

Proof: Observe that a chosen number changes only after exiting from the CS, and that a number is ≠ 0 iff the corresponding processor is either in the entry (bakery) section or in the CS. Now, since p_i is in the CS, it passed the second wait statement for j=k.

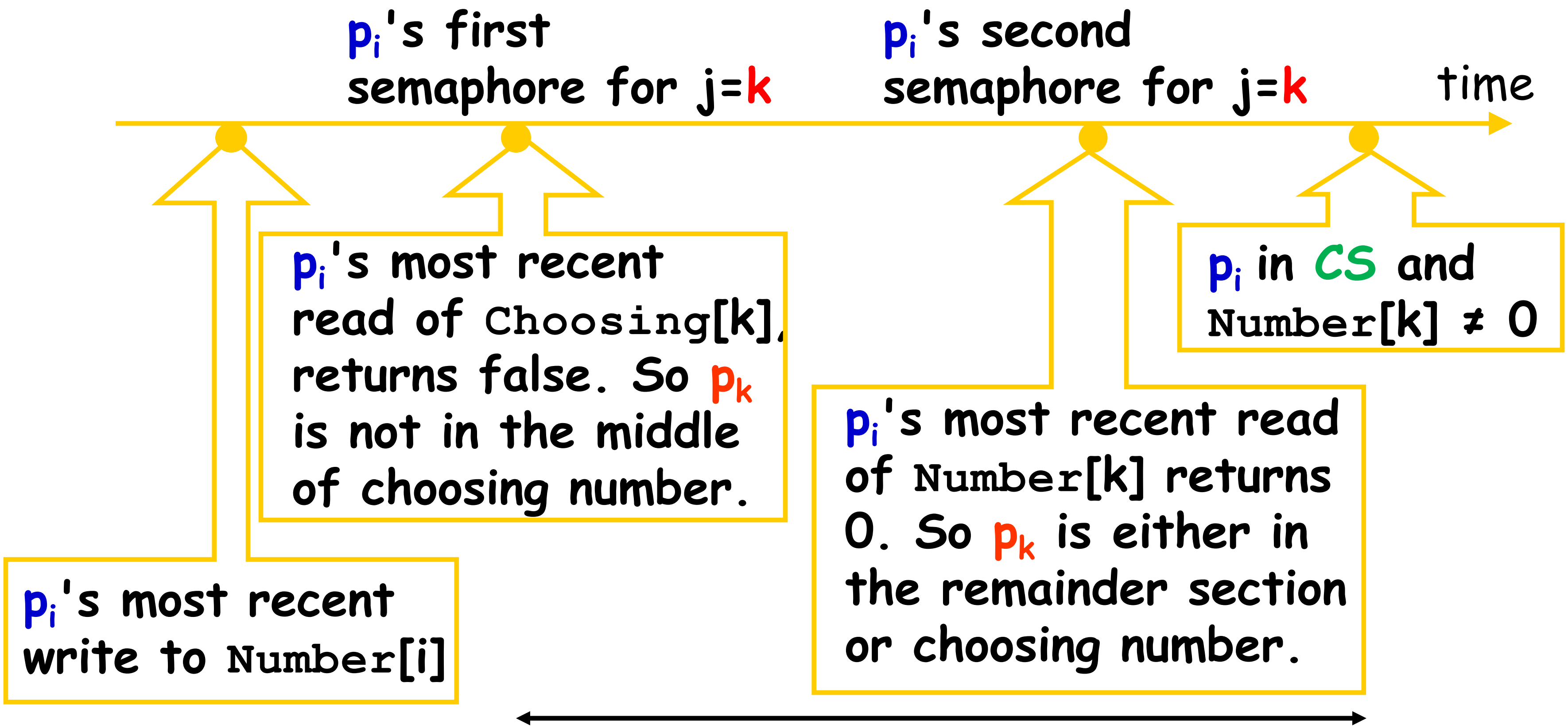
There are two cases:

p_i's most recent read of `Number`[k] (second semaphore)

p_i in CS and `Number`[k] ≠ 0

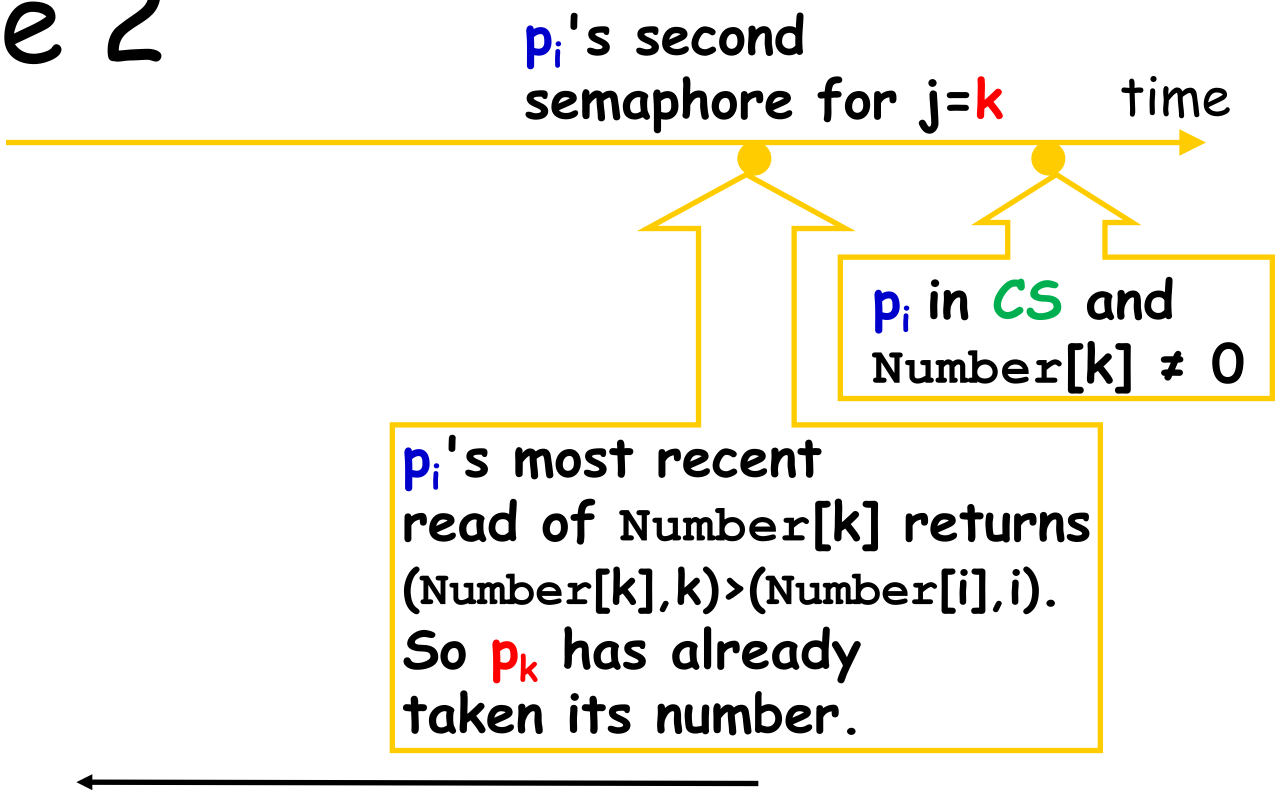
Case 1: returns 0
Case 2: returns (`Number`[k],k)>(`Number`[i],i)

Case 1



So p_k chooses its number in this interval, sees p_i's number, and then chooses a larger one (i.e., Number[k] > (Number[i]); so, it will never enter in CS before than p_i, which means that its number does not change all over the time p_i is in the CS, and so the claim is true

Case 2



So p_k chooses Number[k]≥Number[i] in this interval, and does not change it until p_i exits from the CS, since it cannot surpass p_i. Indeed, p_k will be stopped by p_i either in the first wait statement (in case p_k finished its choice before than p_i and p_i is still choosing its number – in this case, Number[k]=Number[i]), or in the second one (since (Number[i],i)<(Number[k],k)). Thus, it will remain (Number[i],i)<(Number[k],k) until p_i finishes its CS, and the claim follows.

END of PROOF

Mutual Exclusion for BA

- **Mutual Exclusion:** Suppose p_i and p_k are simultaneously in CS, $i \neq k$.
  - By Lemma 1, both have number > 0.
  - Since `Number`[k], `Number`[i] $\neq$ 0, by Lemma 2
    - (`Number`[k],k) > (`Number`[i],i) and
    - (`Number`[i],i) > (`Number`[k],k)

Contradiction!

No Lockout for BA

- Assume in contradiction there is a starved processor.
- Starved processors are stuck at the semaphores, *not* while choosing a number.
- Starved processors can be stuck only at the second semaphore, since sooner or later the `Choosing` variable of each processor will become false
- Let p_i be a starved processor with smallest (`Number`[i],i).
- Any processor entering entry section after that p_i chose its number, will choose a larger number, and therefore cannot surpass p_i
- Every processor with a smaller ticket eventually enters *CS* (not starved) and exits, setting its number to 0. So, in the future, its number will be either 0 or larger than `Number`[i]
- Thus p_i cannot be stuck at the second semaphore forever by another processor.

Contradiction!

What about bounded waiting?

YES: It is easy to see that any processor in the entry section can be surpassed at most once by any other processor (and so in total it can be surpassed at most $n-1$ times).

Space Complexity of BA

- Number of shared variables is 2n
- `Choosing` variables are booleans
- `Number` variables are unbounded: as long as the CS is occupied and some processor enters the entry section, the ticket number increases
- Is it possible for an algorithm to use less shared space?

Bounded-space 2-Processor Mutex Algorithm with no deadlock

- Start with a bounded-variables algorithm for 2 processors with no deadlock, then extend to no lockout, then extend to n processors.
- Use 2 binary shared read/write variables (intuition: if p_i wants to enter into the CS, then it sets `W[i]` to 1):
  - `W[0]`: initially 0, written by p_0 and read by p_1
  - `W[1]`: initially 0, written by p_1 and read by p_0
- Asymmetric (or non-homogenous) code: p_0 always has priority over p_1

Bounded-space 2-Processor Mutex Algorithm with no deadlock

Code for p_0's entry section:

```
.
.
W[0] := 1
.
.
wait until W[1] = 0
```

Semaphore

Code for p_0's exit section:

```
.
W[0] := 0
```

Bounded-space 2-Processor Mutex Algorithm with no deadlock

Code for p_1 's entry section:

```
1   W[1] := 0
2   wait until W[0] = 0
3   W[1] := 1
4   .
5      if (W[0] = 1) then goto Line 1
6   .
```

Semaphore

Code for p_1 's exit section:

```
7   .
8   W[1] := 0
```

Analysis

- Satisfies mutual exclusion: processors use `W` variables to make sure of this (a processor enters only when its own `W` variable is set to 1 and the other `W` variable **is seen** to be 0; notice that when p_1 is in the CS and p_0 is waiting at Line 5 in the entry, then both `W[0]` and `W[1]` are equal to 1, while if p_0 is in the CS and p_1 is waiting at Line 2 in the entry, then `W[0]=1`, while `W[1]=0`)
- Satisfies no-deadlock: if p_0 wants to enter, it cannot be locked by p_1 (since p_1 will be forced to set `W[1]`:=0)
- But unfair w.r.t. p_1 (it can remain locked, if p_0 sets `W[0]` to 1 continuously between line 3 and 5 of p_1 execution)

$\Rightarrow$ Fix it by having the processors alternate in having the priority

Bounded-space 2-Processor Mutex Algorithm with no lockout

Uses 3 binary shared read/write variables and is symmetric:

- `W[0]`: initially 0, written by p_0 and read by p_1
- `W[1]`: initially 0, written by p_1 and read by p_0
- `Priority`: initially 0, written and read by both

Bounded-space 2-Processor Mutex Algorithm with no lockout

Code for p_i's entry section:

```
1   W[i] := 0
2   wait until W[1-i] = 0 or Priority = i
3   W[i] := 1
4   if (Priority = 1-i) then
5     if (W[1-i] = 1) then goto Line 1
6   else wait until (W[1-i] = 0)
```

Semaphores

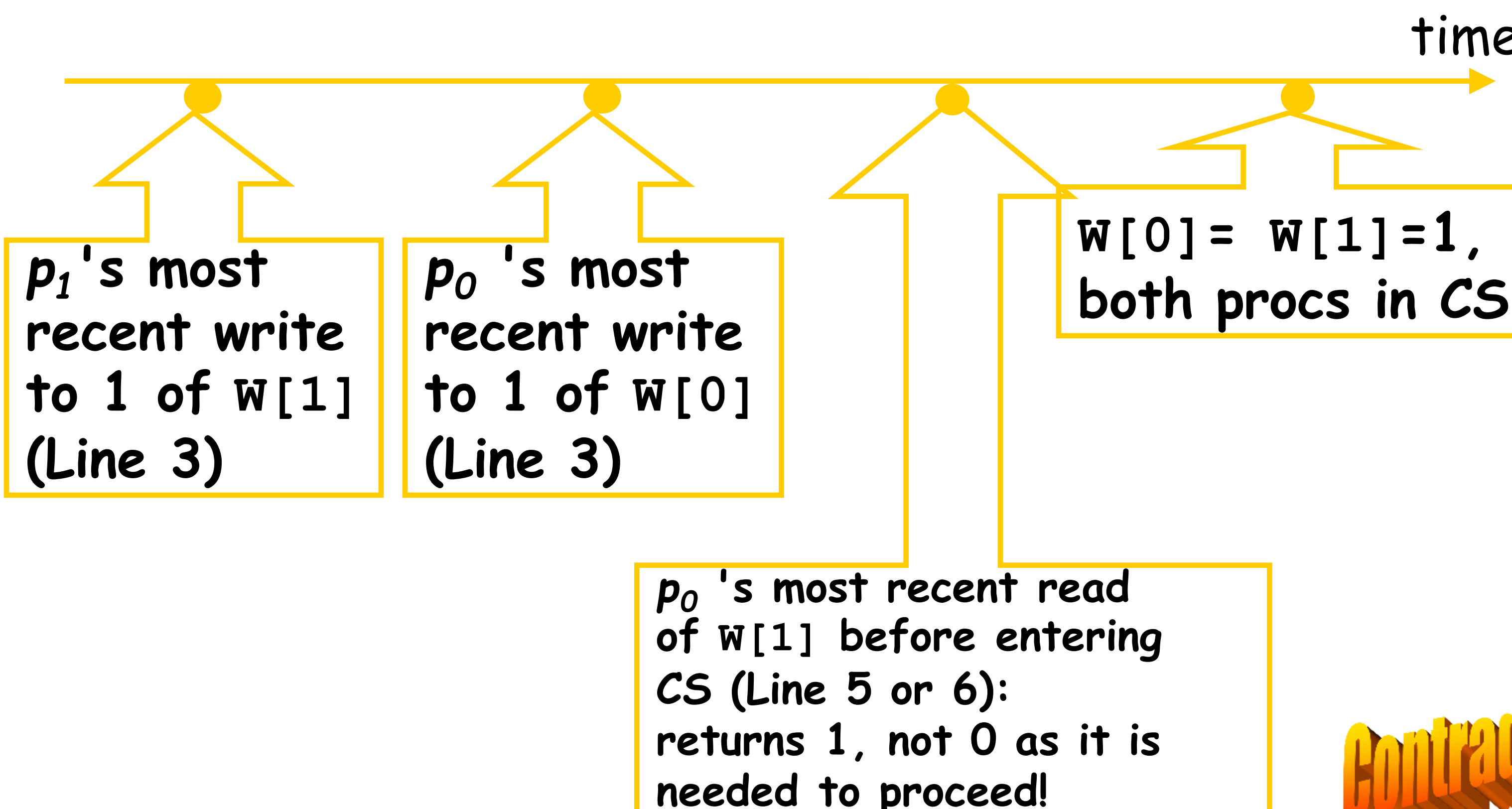
Code for p_i's exit section:

```
7   Priority := 1-i
8   W[i] := 0
```

Analysis: ME

Mutual Exclusion:

- Suppose in contradiction p_0 and p_1 are simultaneously in CS.
- W.l.o.g., assume p_1 last write of `W[1]` before entering CS happens **not later** than p_0 last write of `W[0]` before entering CS



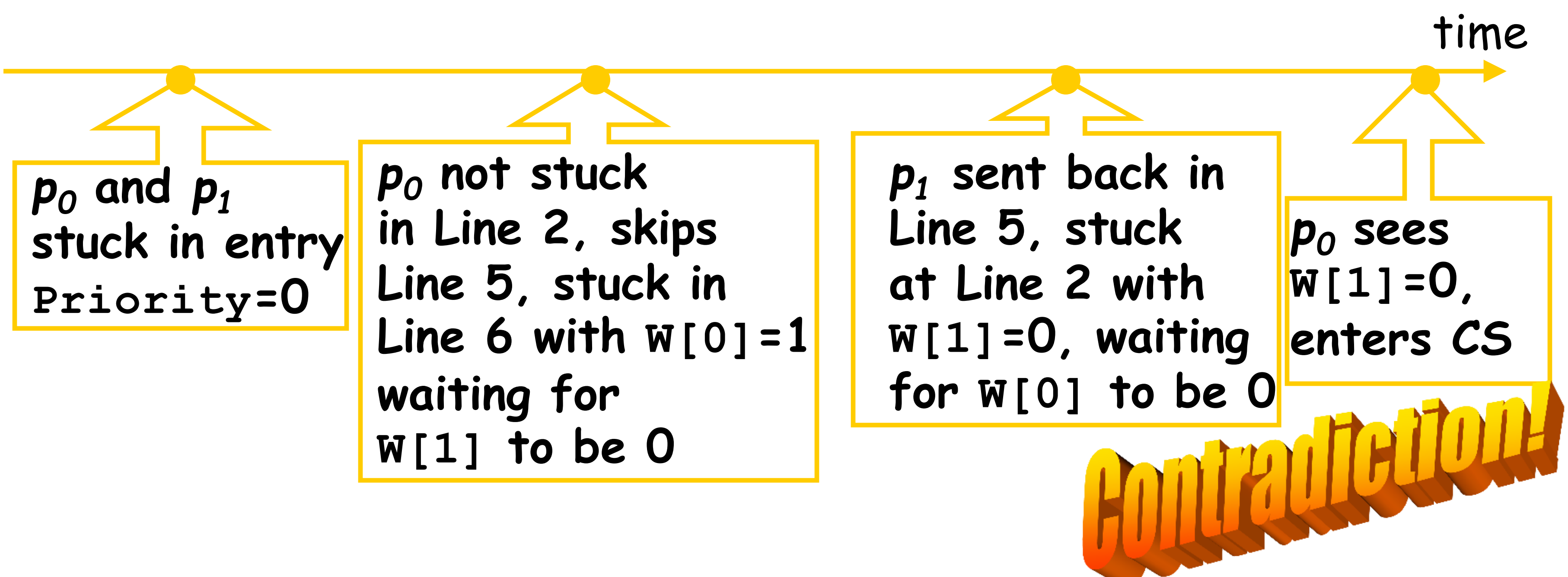
Contradiction!

Analysis: No-Deadlock

- Useful for showing no-lockout.
- If one processor ever stays in the remainder section forever, the other one cannot be starved.
  - Ex: If p_1 enters remainder forever, then p_0 will keep seeing `W[1]` = 0.
- So any deadlock would starve both processors in the entry section

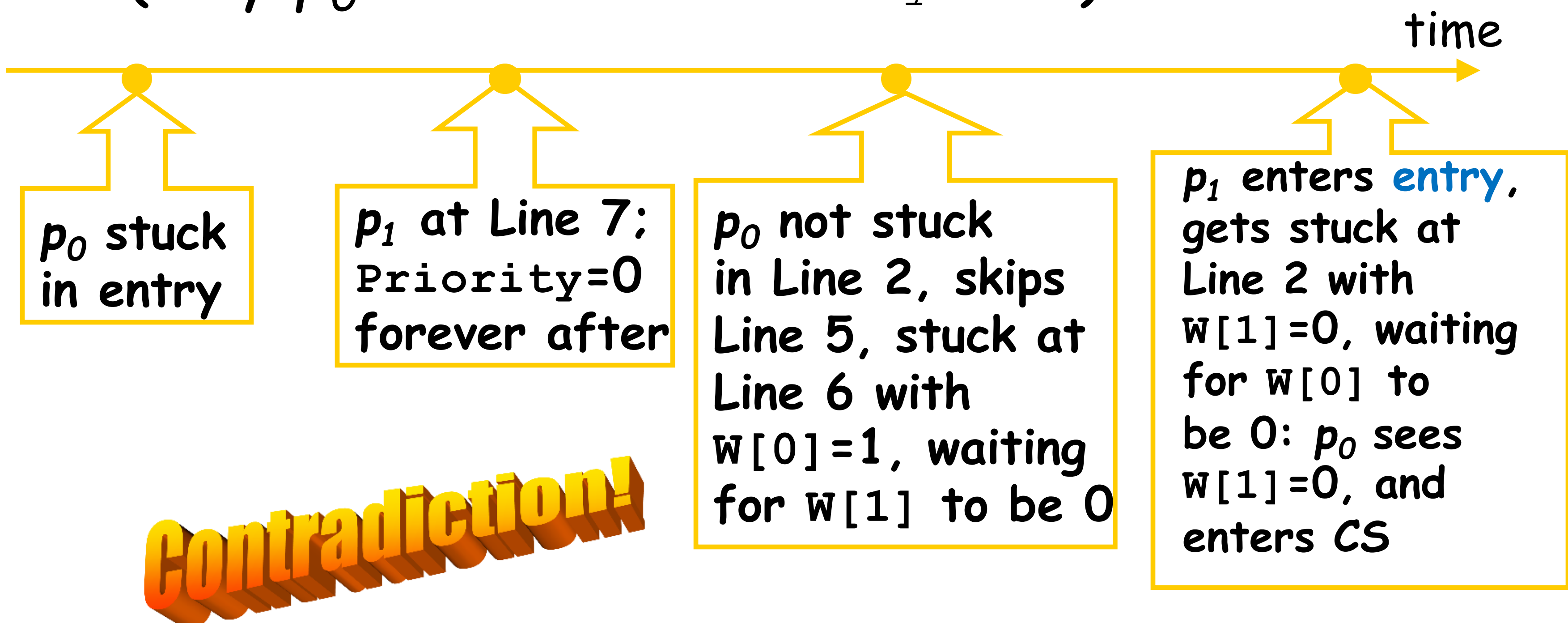
Analysis: No-Deadlock

- Suppose in contradiction there is deadlock, and w.l.o.g., suppose `Priority` gets stuck at 0 after both processors are stuck in their entry sections (indeed `Priority` cannot be changed within the entry section):



Analysis: No-Lockout

- Suppose in contradiction p_0 is starved.
- Since there is no deadlock, p_1 enters CS infinitely often.
- The first time p_1 executes Line 7 in exit section after p_0 is stuck in entry, `Priority` gets stuck at 0 (only p_0 can set `Priority` to 1)



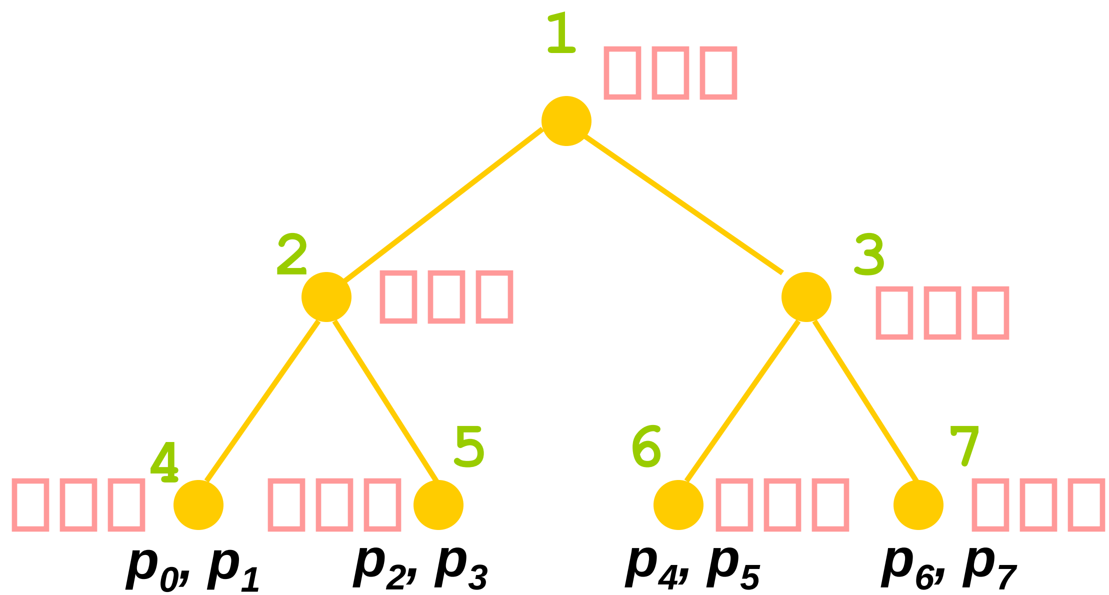
Bounded Waiting?

- NO: A processor, even if having priority, might be surpassed repeatedly (in principle, an unbounded number of times) when it is in between Line 2 and 3.

Bounded-space n-Processor Mutex Algorithm with no lockout

- Can we get a bounded-space no-lockout mutex algorithm for $n>2$ processors?
- Yes! For the sake of simplicity, assume that $n=2^k$, for some $k>1$.
- Based on the notion of a tournament tree: complete binary tree with $n-1$ nodes
  - tree is conceptual only! does not represent message passing channels
- A copy of the 2-processor algorithm is associated with each tree node
  - includes separate copies of the 3 shared variables

Tournament Tree



We label the tree nodes from top to down and from left to right, from 1 to $n-1$; it is not hard to see that, by construction, processor p_i, $i=0,\dots, n-1$, remains associated with node labelled $2^{k-1} + \lfloor i/2 \rfloor$, where $k = \log n$ (recall that $n=2^k$). Notice that, in general, if $n \neq 2^k$, then we complete the tree by adding "dummy" leaves

Tournament Tree Mutex Algorithm

- Each processor begins entry section at the associated leaf (2 processors per leaf)
- *A* processor proceeds to next level in the tree by winning the 2-processor competition for current tree node:
  - on left side, plays role of p_0
  - on right side, plays role of p_1
- When a processor wins the 2-processor algorithm associated with the tree root, it enters CS.

The code

procedure Node(v: integer; $side$: 0..1)

L: $want^v_{side} := 0$

wait until ($want^v_{1-side} = 0$ or $priority^v = side$)

$want^v_{side} := 1$

Entry

if ($priority^v = 1 - side$) then

  if ($want^v_{1-side} = 1$) then goto L

else wait until ($want^v_{1-side} = 0$)

if ($v = 1$) then   /* at the root */

  ⟨Critical Section⟩

Exit

  else Node($\lfloor v/2 \rfloor$, v mod 2)

$want^v_{side} := 0$

$priority^v := 1 - side$

end procedure

More on TT Algorithm

- Code is recursive
- p_i begins at tree node $2^{k-1} + \lfloor i/2 \rfloor$, playing role of $p_{i \bmod 2}$, where $k = \log n$.
- After winning at node v, "critical section" for node v is
  - entry code for all nodes on path from $\lfloor v/2 \rfloor$ to root
  - real critical section
- Finally, executes exit code for all nodes on path from root to v (in each of these nodes, gives *priority* to the other side and sets its *want* variable to 0)

Analysis

- **Correctness**: based on correctness of 2-processor algorithm and tournament structure:
  - Mutual exclusion for TT algorithm follows from ME for 2-processors algorithm at tree root.
  - No-lockout for tournament algorithm follows from no-lockout for the 2-processor algorithms at all nodes of tree
- **Space Complexity:** 3(n-1) boolean read/write shared variables.
- **Bounded Waiting?** No, as for the 2-processor algorithm.

Homework

Consider the mutex problem on a synchronous DS of 8 processors (with ids in 0..7). Show an execution of the tournament tree algorithm by assuming the following:

1. Initially, all the *want* and *priority* variables are equal to 0;
2. The system is totally synchronous, i.e., lines of code are executed simultaneously by all processors;
3. Throughout the entry section, a processor ends up a round either if it wins the competition (and possibly it enters the CS), or if it executes 7 lines of codes;
4. If a node enters the CS at round k, then it exits at round k+1;
5. Throughout the exit section, a processor ends up a round after having executed the exit code for a node of the tree;
6. p_0, p_1, p_3, p_5 and p_6 decide to enter the CS in round 1, while the remaining processors decide to enter the CS in round 2.

Hints: 16 rounds until the last processor completes the exit section; entering sequence is p_0, p_5, p_3, p_6, p_1, p_4, p_2, p_7