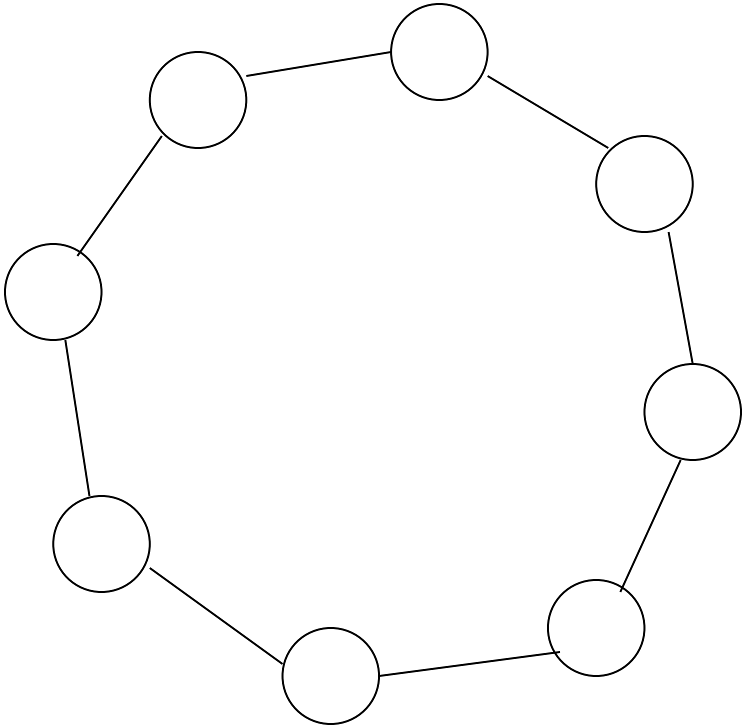
Algorithms for COOPERATIVE DS:
Leader Election in the MPS model

Leader Election (LE) problem

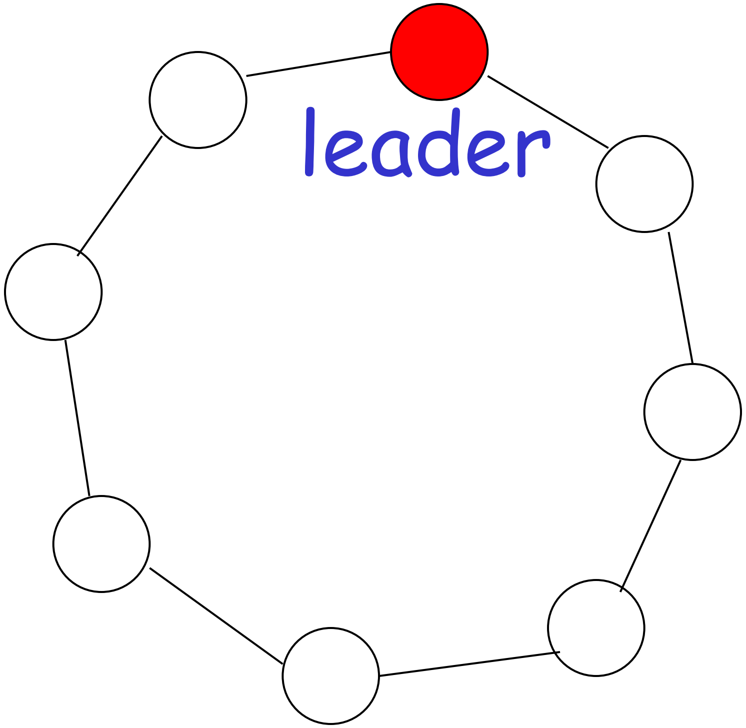
- In a DS, it is often needed to designate a single processor (i.e., a leader) as the coordinator of some forthcoming task (e.g., finding a spanning tree of a graph using the leader as the root)
- In a LE computation, each processor must decide between two internal states: either **elected** (won), or **not-elected** (lost, default state).
- Once an elected state is entered, processor will remain forever in an elected state: i.e., irreversible decision
- **Correctness**: In every admissible execution, exactly one processor (the **leader**) must enter in the elected state

Leader Election in Ring Networks

Initial state
(all not-elected)



Final state

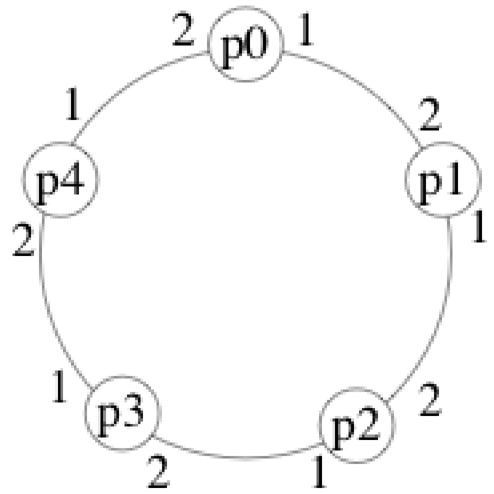


Why Studying Rings?

- Simple starting point, easy to analyze
- Abstraction of a classic LAN topology
- Lower bounds and impossibility results for ring topology also apply to arbitrary topologies

Sense-of-direction in Rings

In an *oriented* ring, processors have a consistent notion of left and right



1 = left = clockwise

2 = right = counter-clockwise

For example, if messages are always forwarded on channel 1, they will cycle clockwise around the ring

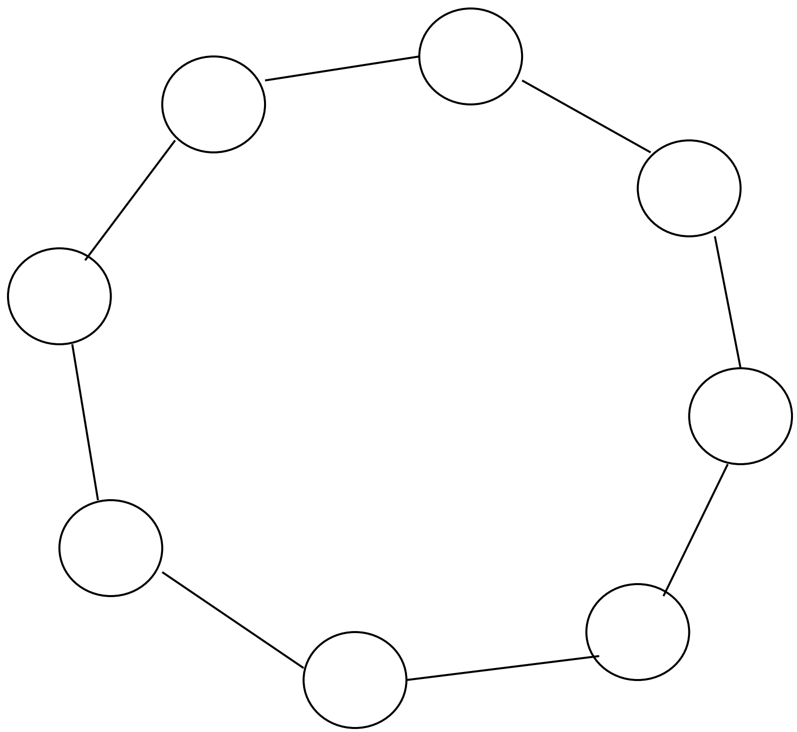
LE algorithms in rings depend on:

Anonymous Ring
Non-anonymous Ring

Size of the network n is known (non-unif.)
Size of the network n is not known (unif.)

Synchronous Algorithm
Asynchronous Algorithm

LE in Anonymous Rings



Every processor runs **exactly** the same algorithm

Every processor does exactly the same execution

Impossibility for Anonymous Rings

Theorem: There is *no* leader election algorithm for anonymous rings, even if

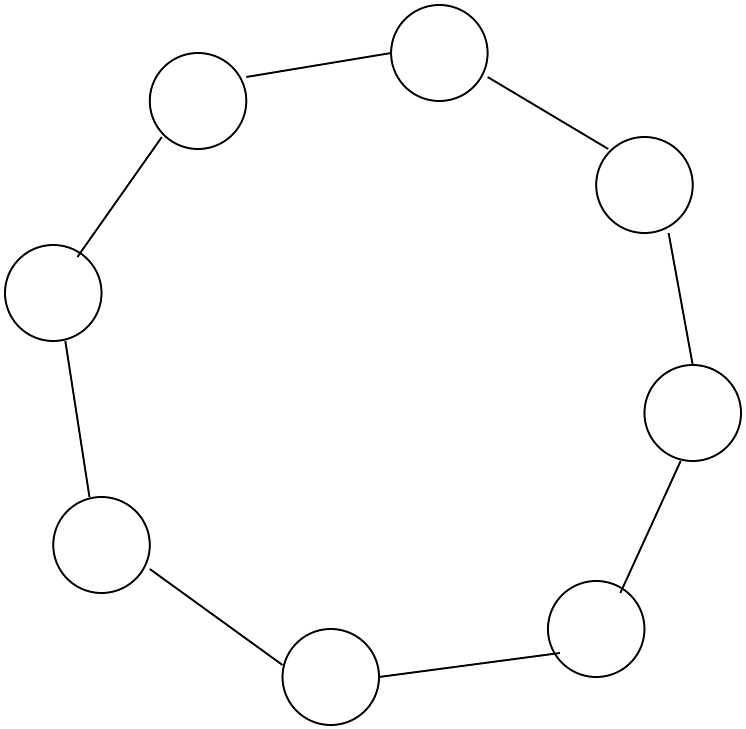
- the algorithm knows the ring size (non-uniform)
- the algorithm is synchronous

Proof Sketch (for non-unif and sync rings): It suffices to show an execution in which an hypothetical algorithm will fail:

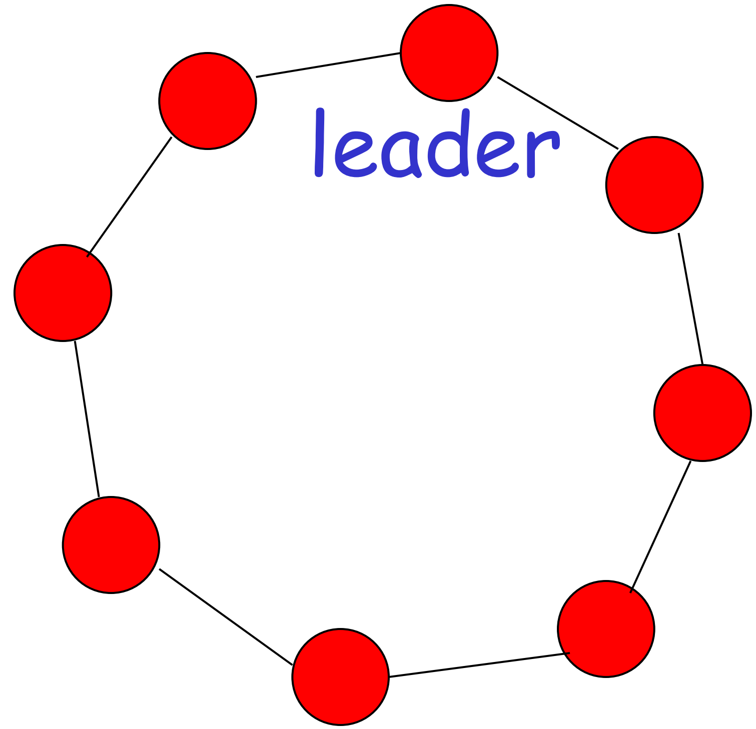
1. Assume all processors wake-up simultaneously (notice this is a worst-case assumption).
2. Every processor begins in the same state (not-elected) with same outgoing msgs, since anonymous (is this one a worst-case assumption?).
3. Every processor receives same msgs, does same state transition, and sends same msgs in round 1.
4. And so on and so forth for rounds 2, 3, …
5. Eventually some processor is supposed to enter an elected state. But then they all would do $\Rightarrow$ uncorrecteness!

Initial state

(all not-elected)



Final state



If one node is elected leader,
then every node is elected leader

Impossibility for Anonymous Rings

Since the theorem was proven for non-uniform and synchronous rings, the same result holds for weaker models:

- uniform
- asynchronous

Rings with Identifiers, i.e., non-anonymous

Assume each processor has a **unique** id.

Don't confuse indices and ids:

- **indices** are 0 to n-1; used only for analysis, not available to the processors
- **ids** are arbitrary nonnegative integers; are available to the processors through local variable *id*.

Overview of LE in Rings with Ids

There exist algorithms when nodes have unique ids.

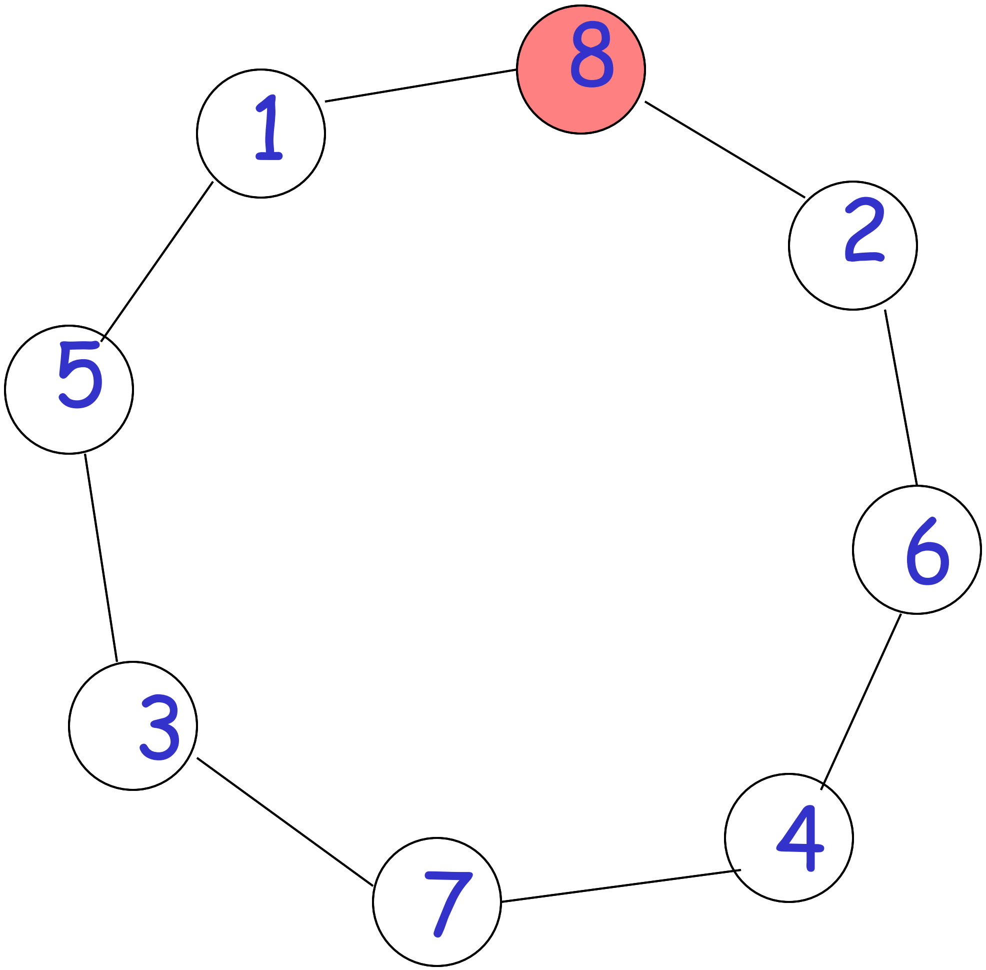
We will evaluate them according to their message (and time) complexity. Best results follow:

- asynchronous rings:
  - $O(n \log n)$ messages
- synchronous rings:
  - $\Theta(n)$ messages, time complexity depending on n and on the magnitude of the identifiers

Above bounds are asymptotically tight (though we will not show lower bounds) and hold for uniform rings.

Asynchronous Non-anonymous Rings

W.l.o.g: the maximum id node is elected leader



An $O(n^2)$ messages asynchronous algorithm: the Chang-Roberts algorithm (1979)

- Every processor which wakes-up (either spontaneously or by a message arrival, no synchronized start is required) sends a message with its own id to the left
- Every processor forwards to the left any message with an id greater than its own id
- If a processor receives its own id it elects itself as the leader, and announces this to the others
- **Remark**: it is uniform (number of processors does not need to be known by the algorithm)
- We will show the algorithm requires $O(n^2)$ messages; we use **O** notation because not all the executions of the algorithm costs n^2, in an asymptotic sense, but only some of them, as we will see

CR algorithm: pseudo-code for a generic processor

```
boolean participant=false;
int leader_id=null;
```

To initiate an election:

```
  send(ELECTION⟨my_id⟩);
  participant:=true;
```

Upon receiving a message ELECTION⟨j⟩:

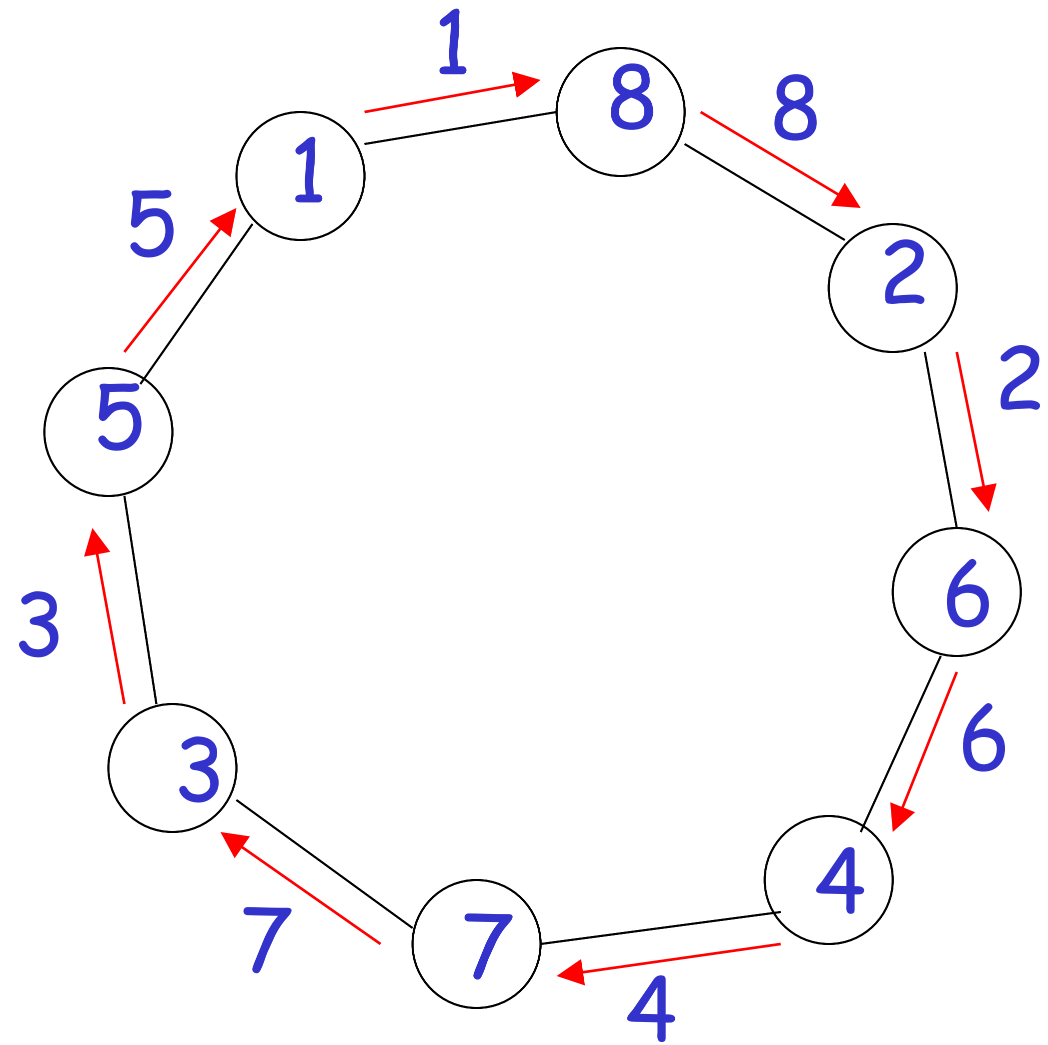
```
  if (j > my_id) then send(ELECTION⟨j⟩);
  if (my_id = j) then send(LEADER⟨my_id⟩);
  if ((my_id > j) ∧ (¬participant)) then
     send(ELECTION⟨my_id⟩);
participant:=true;
```

Upon receiving a message LEADER⟨j⟩:

```
  leader_id:=j;
  if (my_id ≠ j) then send(LEADER⟨j⟩);
```

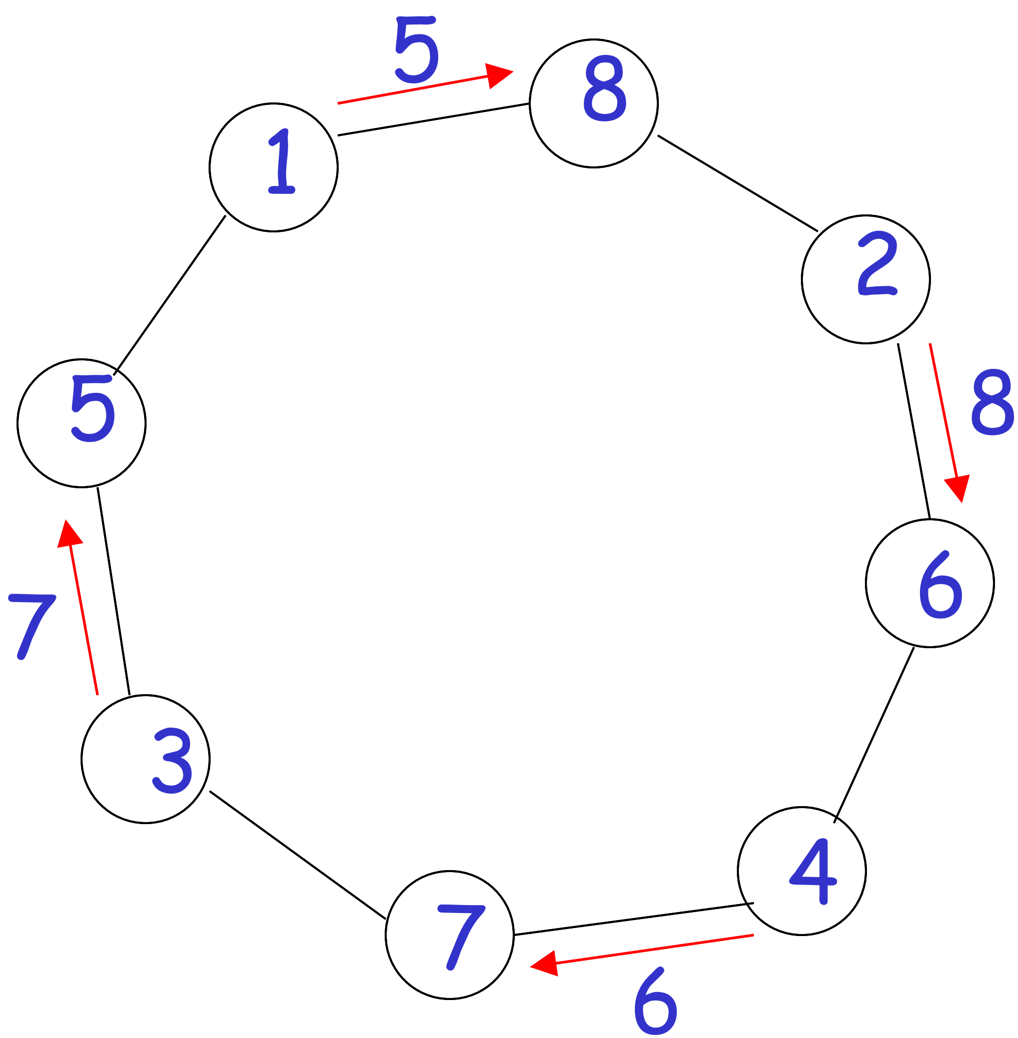
Chang-Roberts algorithm: an execution (all the nodes start together)

Each node sends a message with its id to the left neighbor



If: message received id > current node id

Then: forward message



If: message received id > current node id

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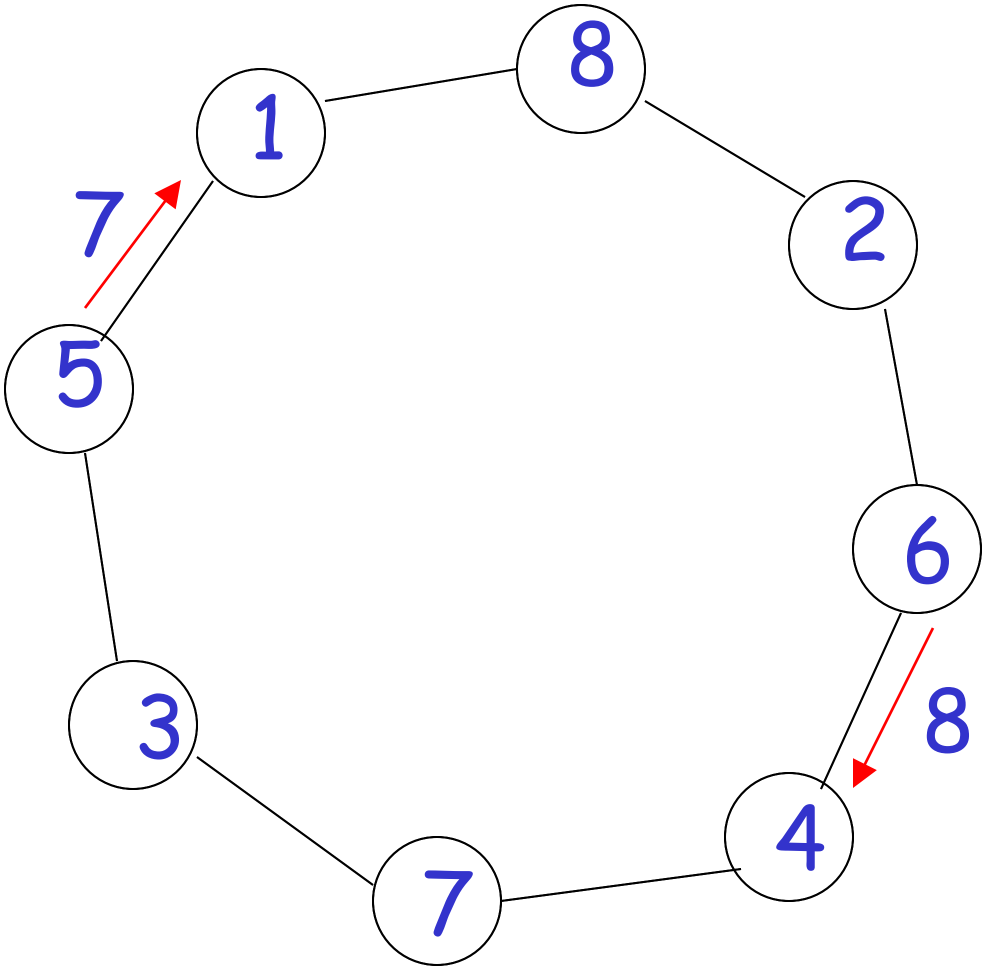

If: message received id > current node id

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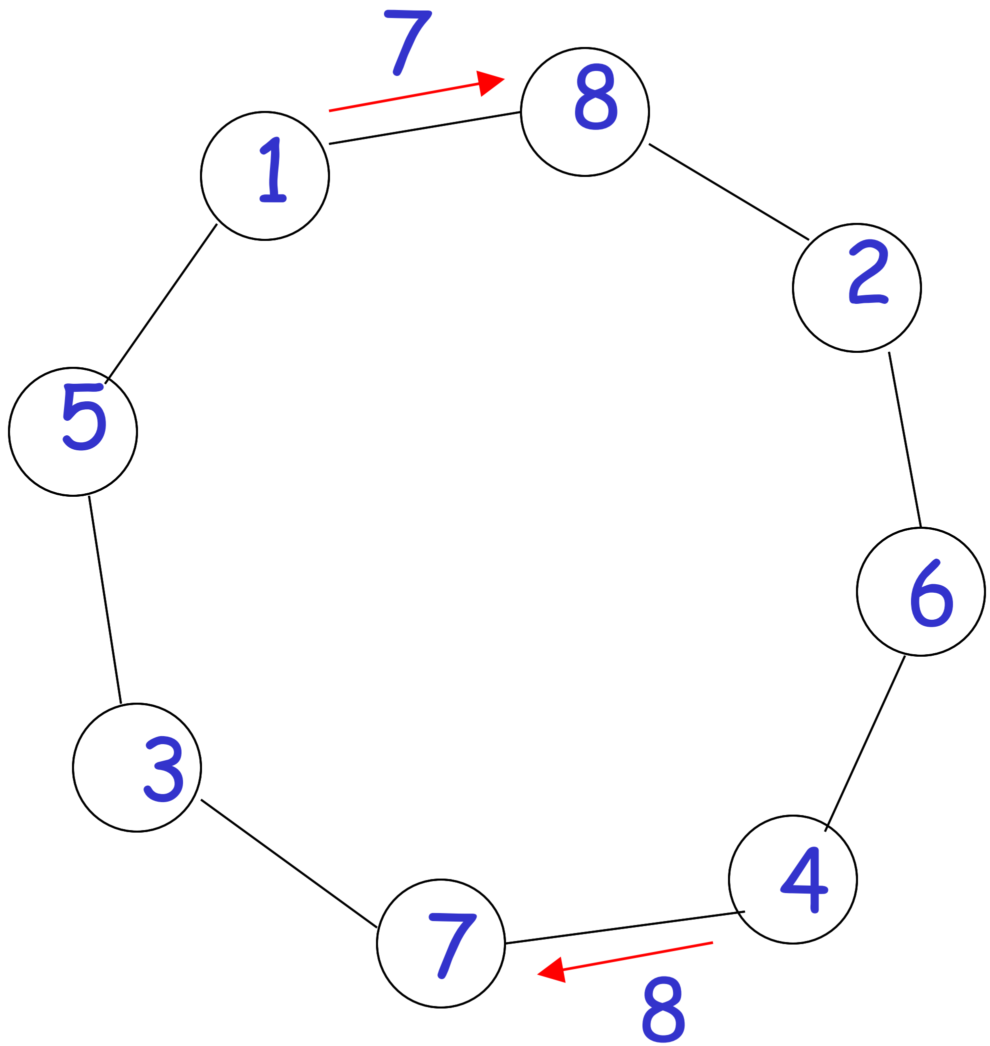
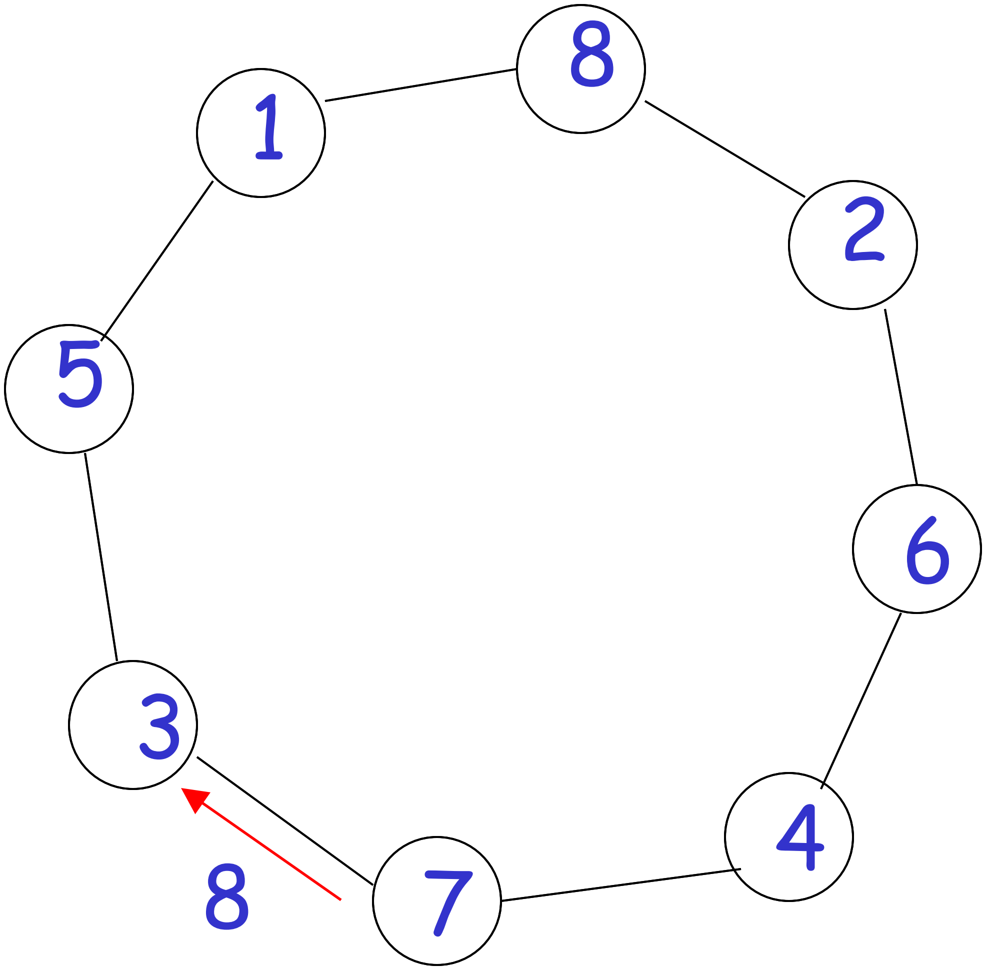


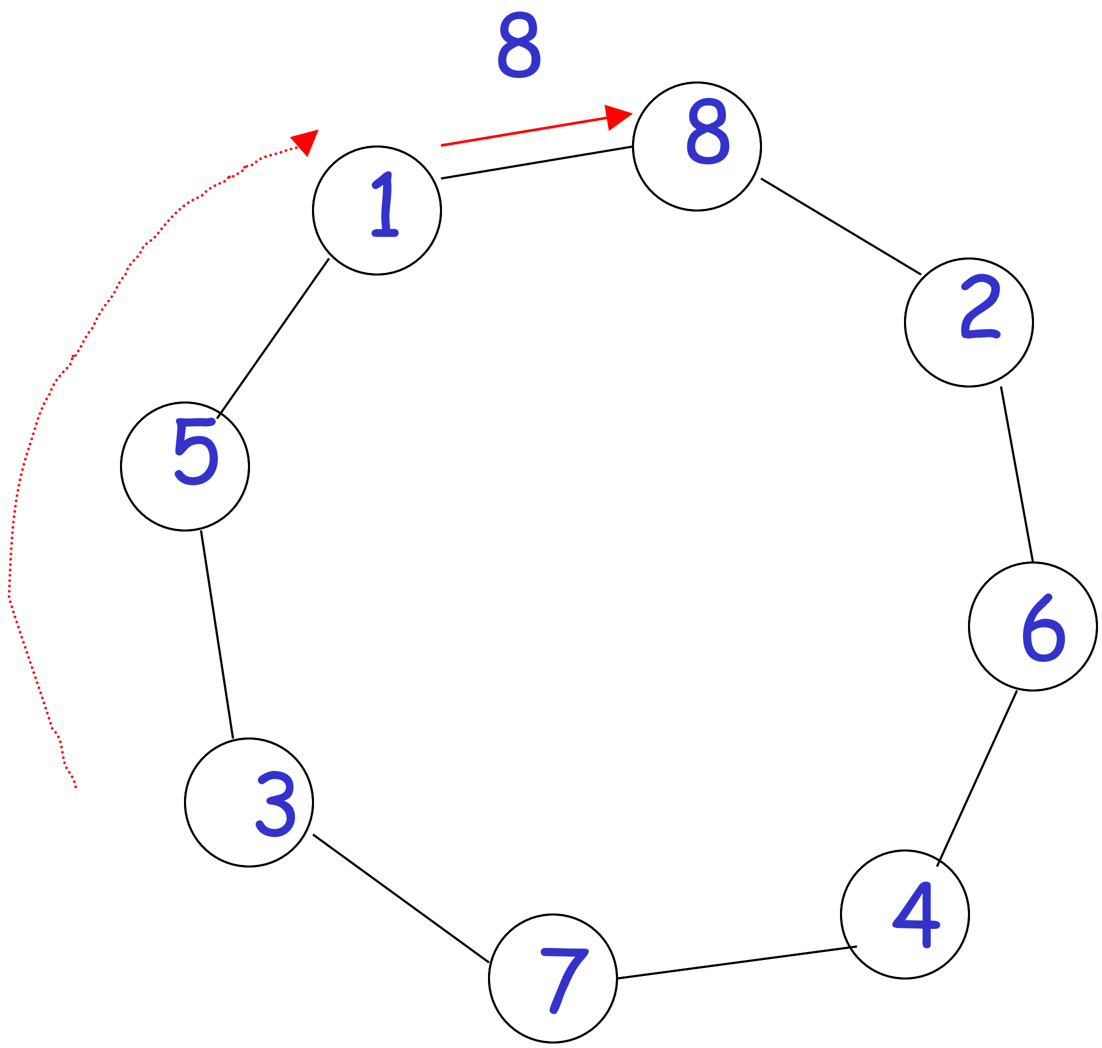
If: message received id > current node id

Then: forward message



If: a node receives its own message

Then: it elects itself a leader



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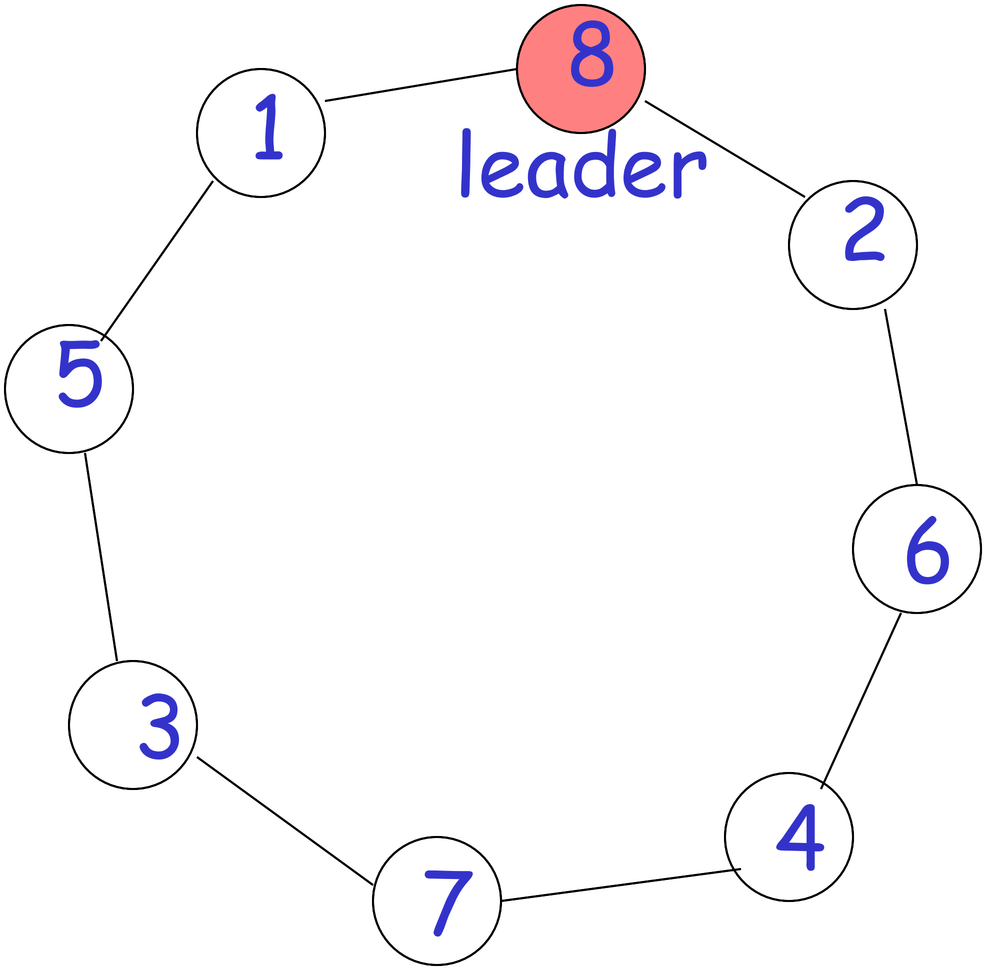


Analysis of Chang-Roberts algorithm

Correctness: Elects processor with largest id.

Indeed, the message containing the largest id passes through every processor, while all other messages will be stopped somewhere

Message complexity: Depends on how the ids are arranged.

largest id travels all around the ring (*n* messages)

2nd largest id travels until reaching largest

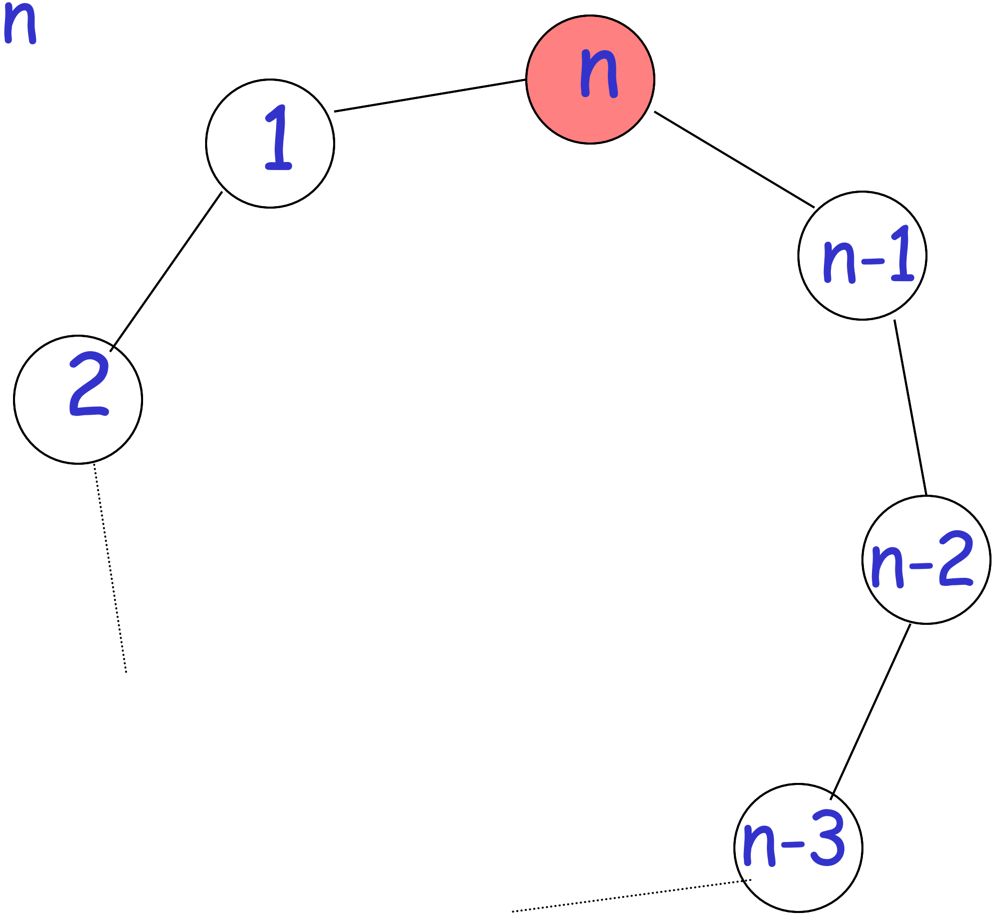
3rd largest id travels until reaching either largest or second largest

etc.

Worst case: $\Theta(n^2)$ messages

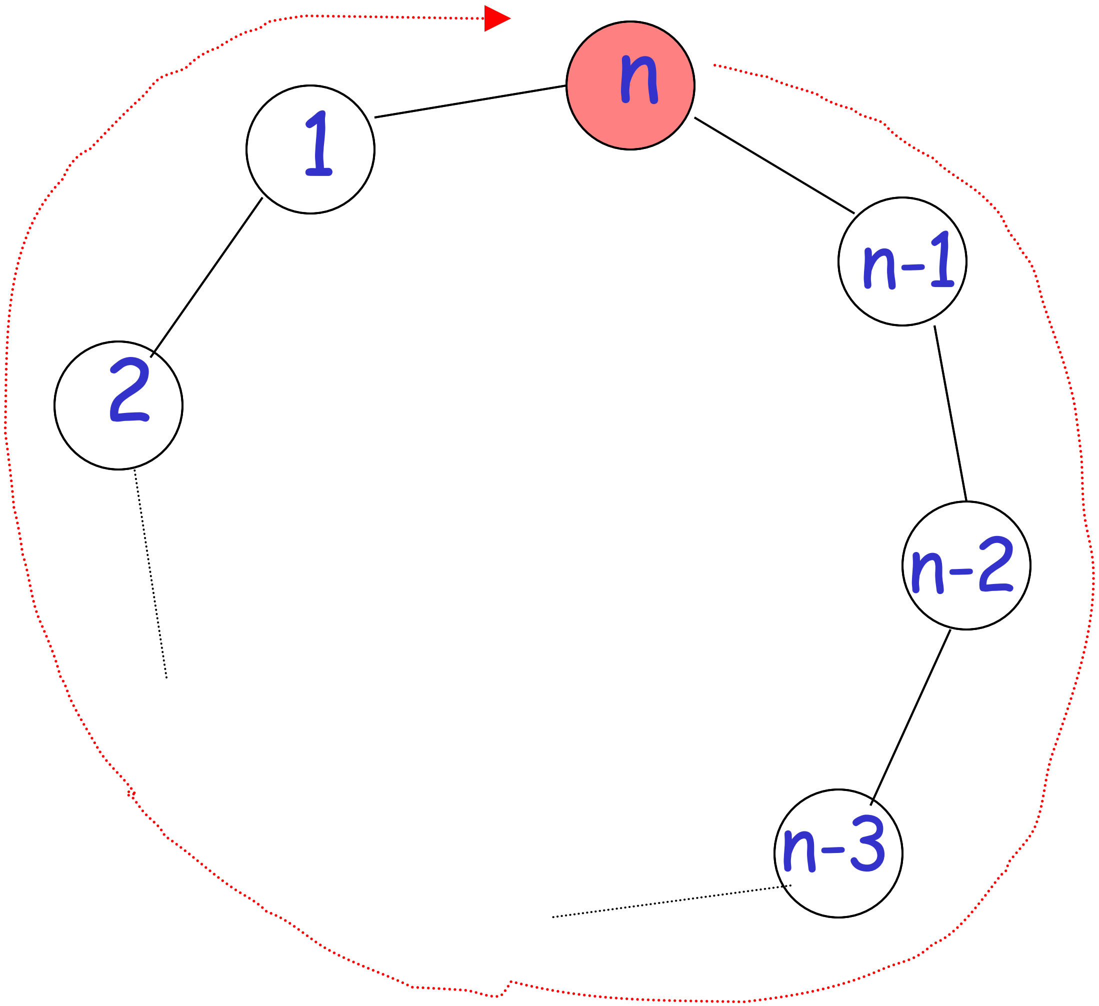
Worst way to arrange the ids is in decreasing order:

- 2nd largest causes *n - 1* messages
- 3rd largest causes *n - 2* messages
- etc.



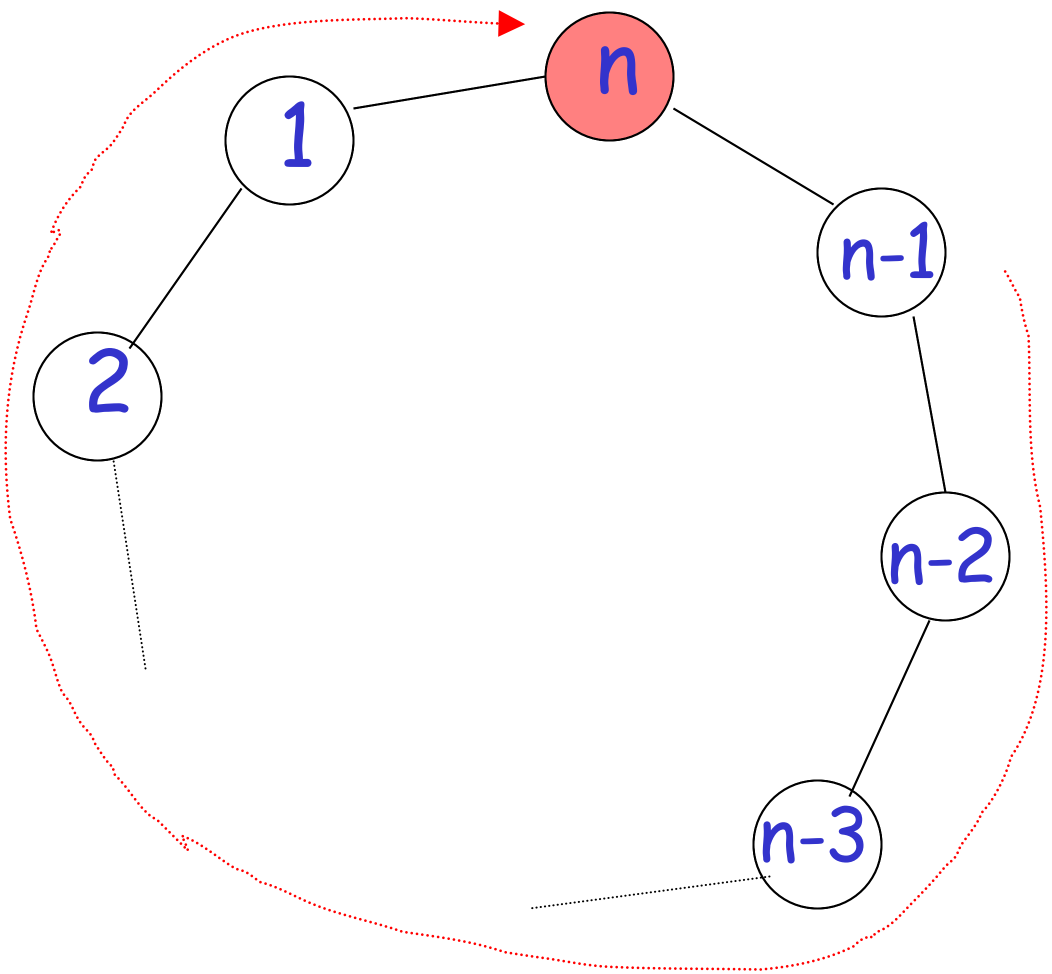
Worst case: $\Theta(n^2)$ messages

n messages



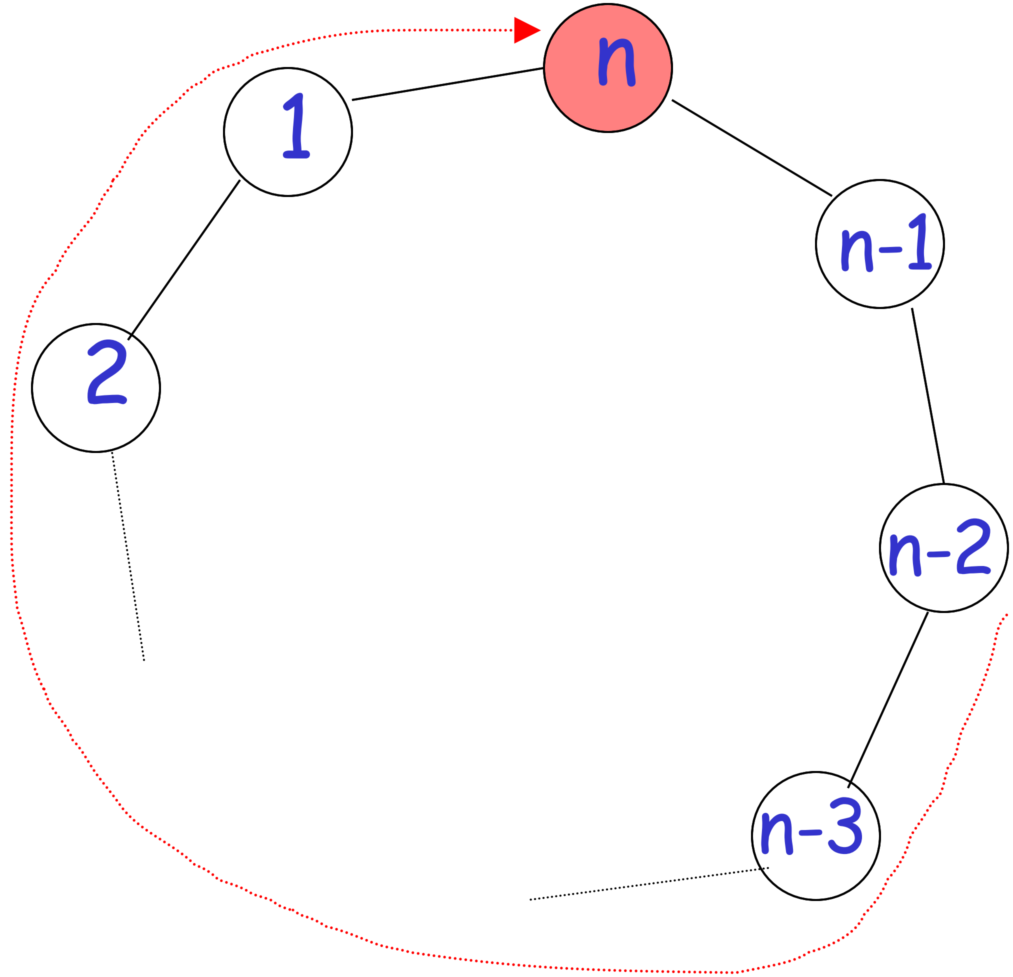
Worst case: $\Theta(n^2)$ messages

n-1 messages



Worst case: $\Theta(n^2)$ messages

n-2 messages



Worst case: $\Theta(n^2)$ messages

Total messages:

$$n+$$

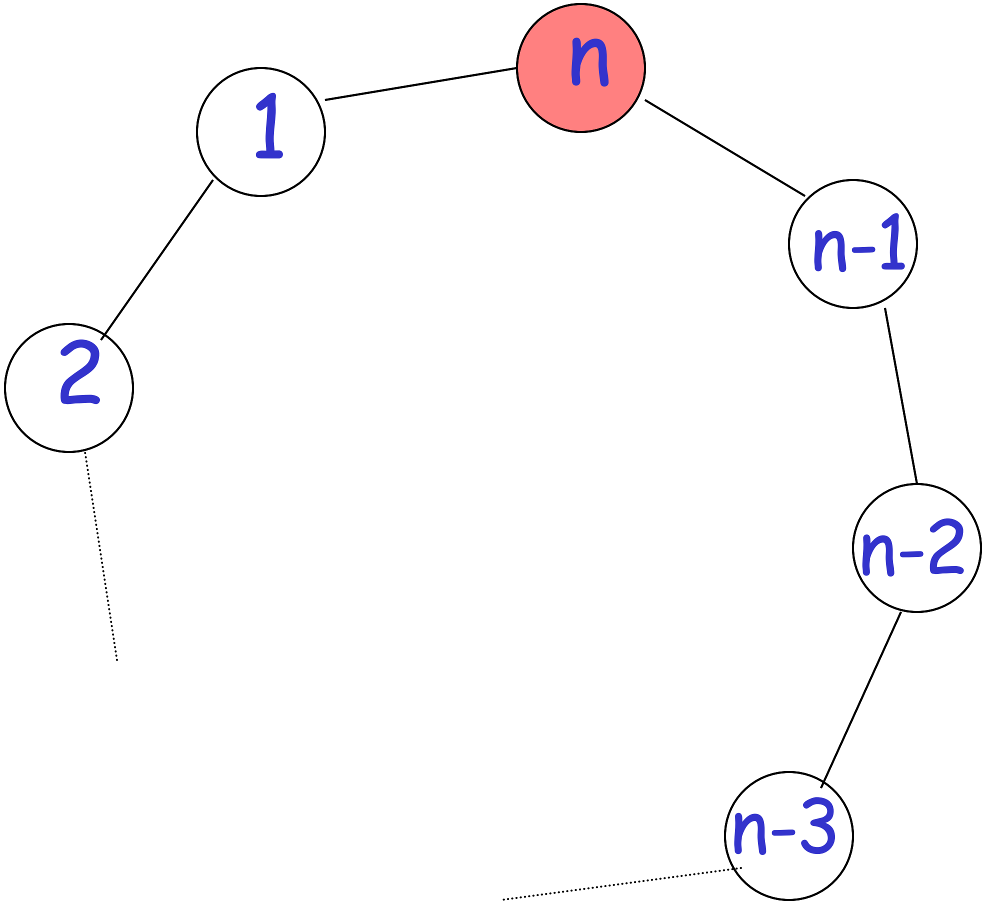
$$n-1+$$

$$n-2+$$

$$\dots$$

$$2+$$

$$1=\Theta(n^2)$$



Best case: Θ(n) messages

Total messages:

$$n+$$

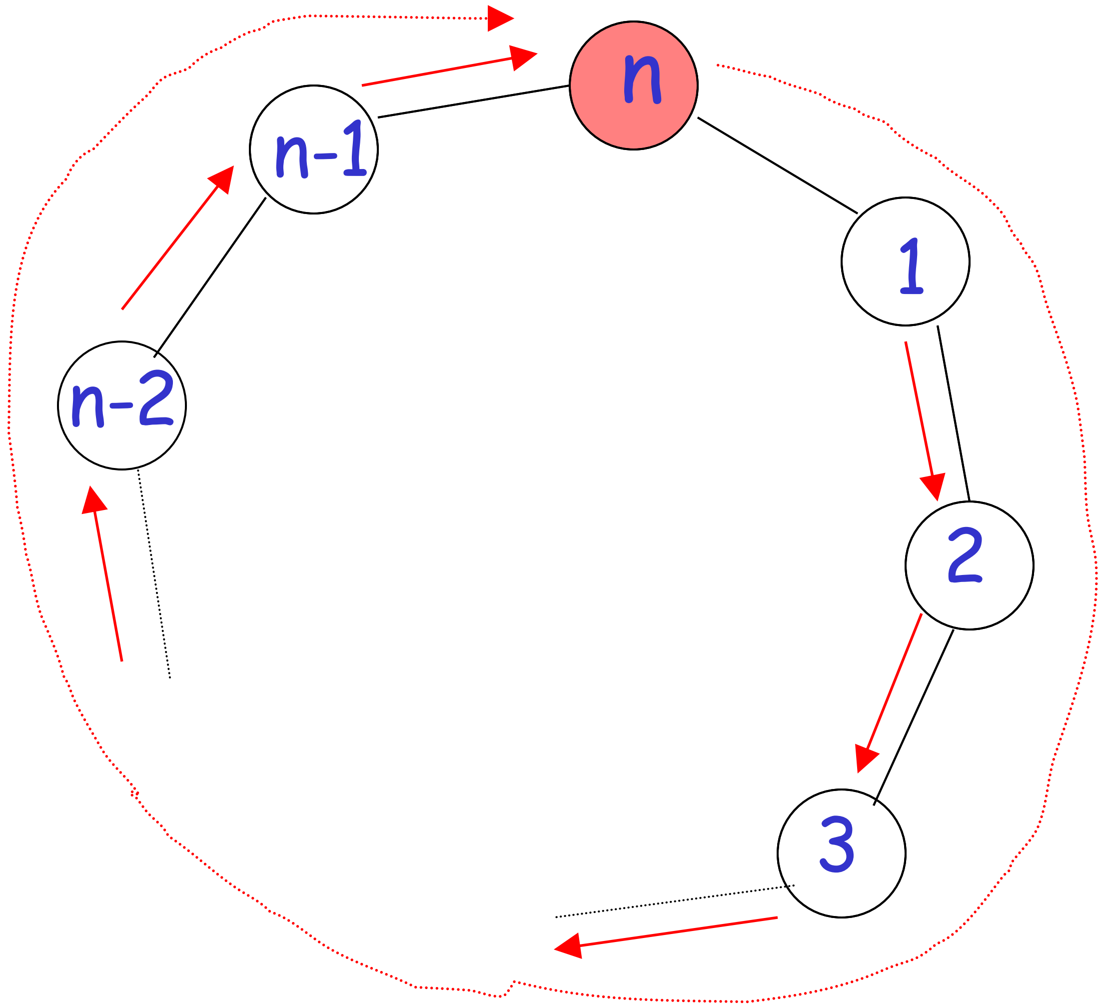
$$1+$$

$$1+$$

$$\ldots$$

$$1+$$

$$1 = \Theta(n)$$



Average case analysis CR-algorithm

Theorem: The average message complexity of the CR-algorithm is $\Theta(n \log n)$.

Sketch of proof: Assume all $n!$ rings (all possible permutations) are equiprobable, and assume that all processors take part to the election (they wake-up simultaneously)

- Probability that a generic id makes exactly 1 step is equal to the probability it makes at least 1 step minus the probability it makes at least 2 steps: Prob(to make exactly 1 step) = 1 – 1/2 = 1/2
- Probability that a generic id makes exactly 2 steps is equal to the probability it makes at least 2 steps minus the probability it makes at least 3 steps: Prob(to make exactly 2 steps) = 1/2 – 1/3 = 1/6
- …
- Probability that a generic id makes exactly k steps is equal to the probability it makes at least k steps minus the probability it makes at least k+1 steps: Prob(to make exactly k steps) = 1/k – 1/(k+1) = 1/k(k+1)
- …
- Probability that a generic id makes exactly n steps is just 1/n

Average case analysis CR-algorithm (2)

$\Rightarrow$ Expected number of steps (i.e., of messages) for each id is

$E(\#\ messages) = \sum_{i=1,\dots,n} i \cdot Prob(\text{to make exactly } i \text{ steps})=$

$= 1\cdot 1/2 + 2\cdot 1/6 + 3\cdot 1/12 + \ . \ . \ . + (n-1)\cdot 1/[n(n-1)] + n\cdot 1/n =$

$= 1/2+1/3+1/4+\dots+1/n+1$, i.e., harmonic series which tends to

$0.69\cdot \log n = \Theta(\log n)$

$\Rightarrow$ Average message complexity is:

$\Theta(n \log n)$ (i.e., $\Theta(\log n)$ for each id) + n (leader announcement)

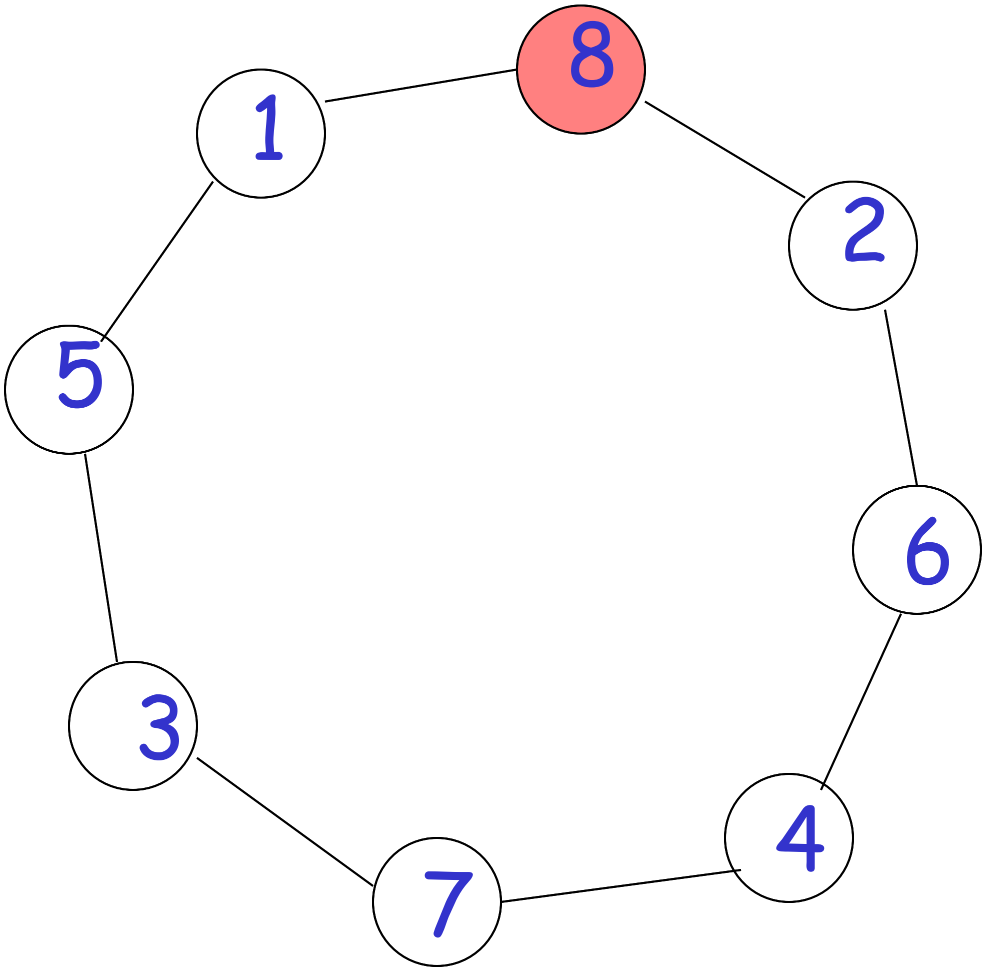
$= \Theta(n \log n)$. □

Can We Use Fewer Messages?

- The $O(n^2)$ algorithm is simple and works in both synchronous and asynchronous model.
- But can we solve the problem with fewer messages?
- Idea:
  - Try to have msgs containing larger ids travel smaller distance in the ring

An O(n log n) messages asyncronous algorithm: the Hirschberg-Sinclair algorithm (1980)

Again, the maximum id node is elected leader



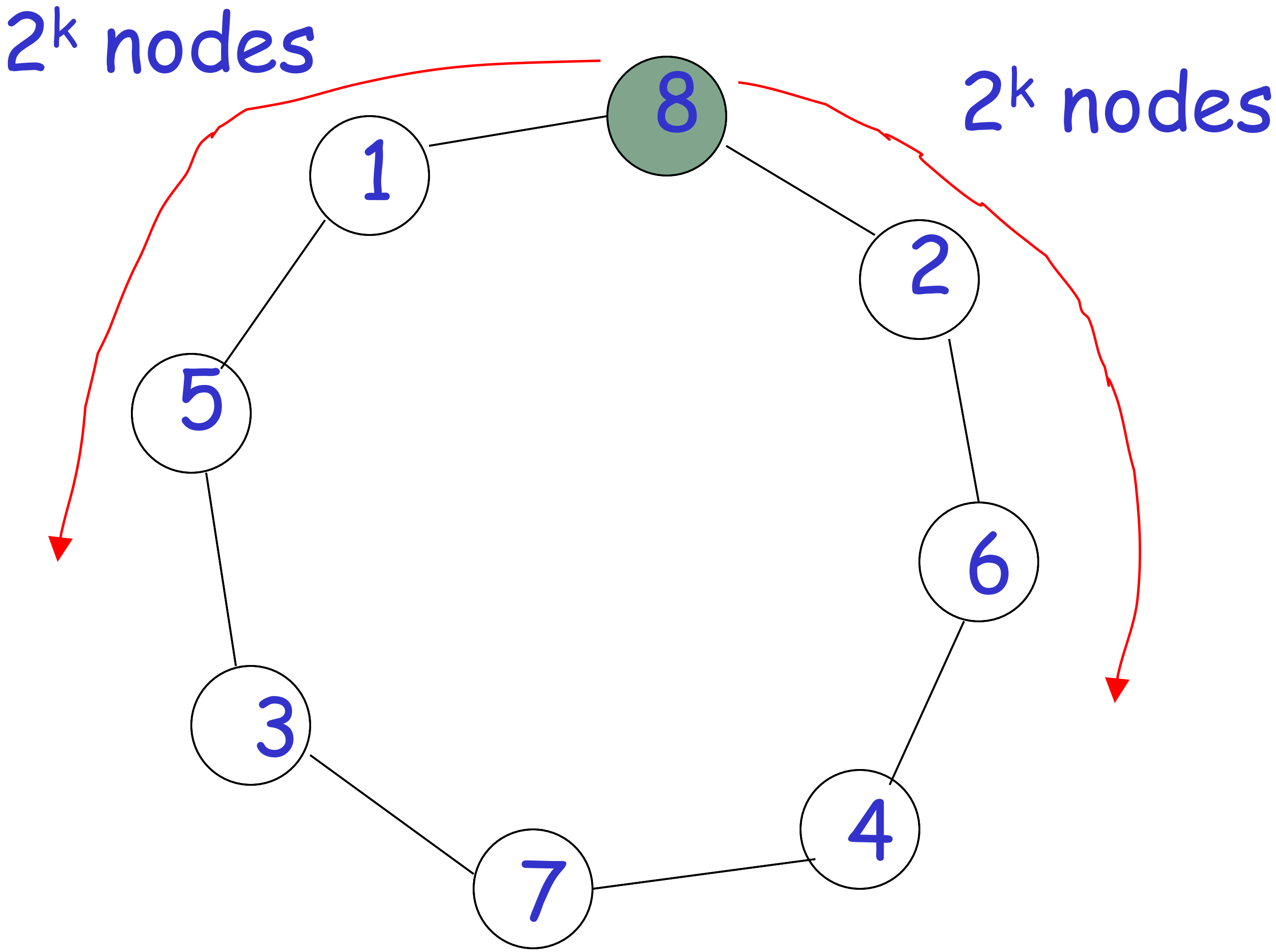
Hirschberg-Sinclair algorithm (1)

- Assume ring is bidirectional
- Carry out elections on increasingly larger sets
- Algorithm works in (asynchronous) phases
- No synchronized start is required: Every processor which wakes-up (either spontaneously or by a message arrival), tries to elect itself as a temporary leader of the current phase to access to the next phase
- p_i becomes a temporary leader in phase k=0,1,2,... iff it has the **largest** id of its 2^k–neighborood, namely of all nodes that are at a distance 2^k or less from it; to establish that, it sends probing messages on both sides
- Probing in phase k requires at most $4 \cdot 2^k$ messages for each processor trying to become leader

Message types

1. Probing (or election) message: it travels from the temporary leader towards the periphery of the actual neighborhood and will contain the fields (id, current phase, step counter); as for the CR-algorithm, a probing message will be stopped if it reaches a processor with a larger id
2. Reply message: it travels from the periphery of the actual neighborhood towards the temporary leader and will contain the fields (id (of the temporary leader), current phase)

2^k −neighborhood



Hirschberg-Sinclair algorithm (2)

- Only processors that win the election in phase k can proceed to phase k+1
- If a processor receives a probe message with its own id, it elects itself as leader
- **Remark**: it is uniform (number of processors does not need to be known by the algorithm)

HS algorithm: pseudo-code for a generic processor

To initiate an election (phase 0):

send(ELECTION$\langle my_id, 0, 1\rangle$) to left and right;

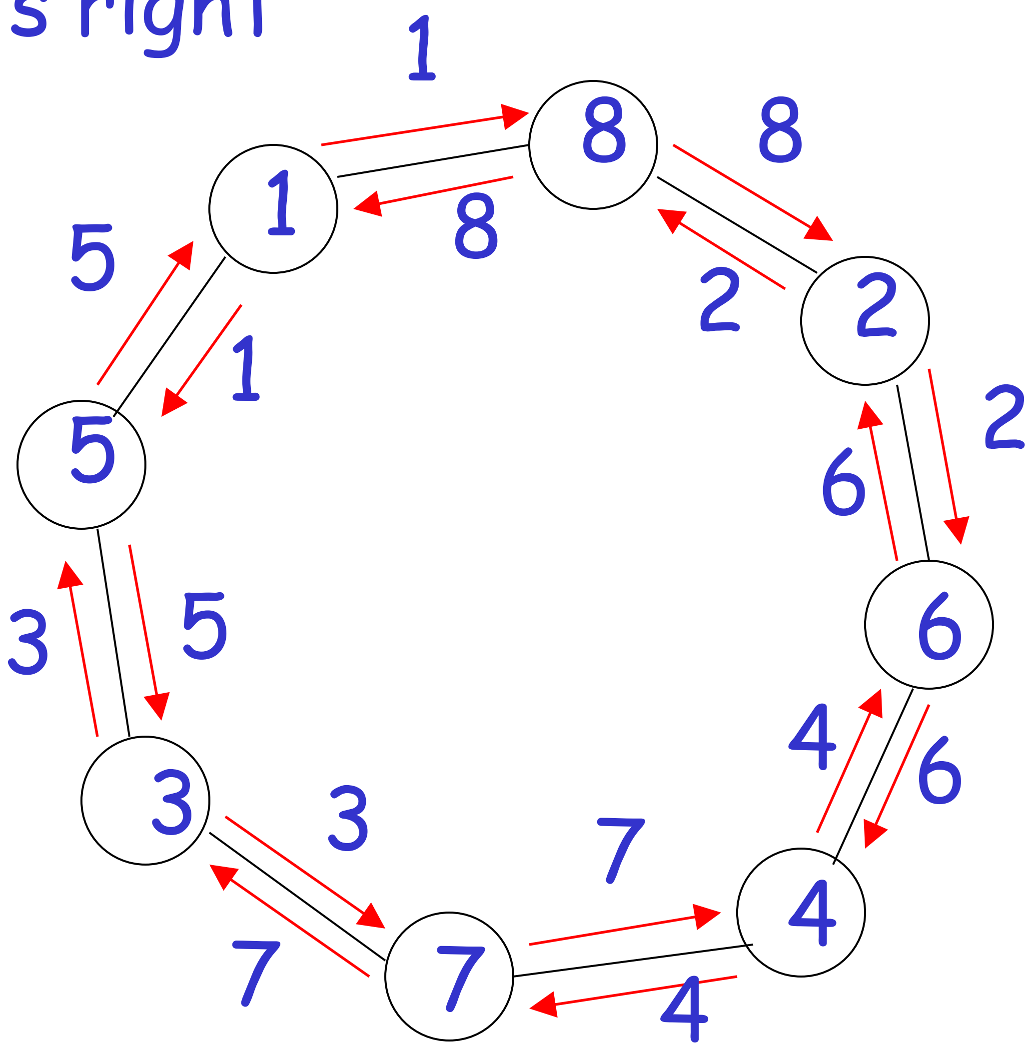
Upon receiving a message ELECTION$\langle j, k, d\rangle$ from left (right):

```
if ((j > my_id) ∧ (d < 2^k)) then
 send(ELECTION⟨j, k, d + 1⟩) to right (left);
if ((j > my_id) ∧ (d = 2^k)) then
 send(REPLY⟨j, k⟩) to left (right);
if (my_id = j) then announce itself as leader;
```

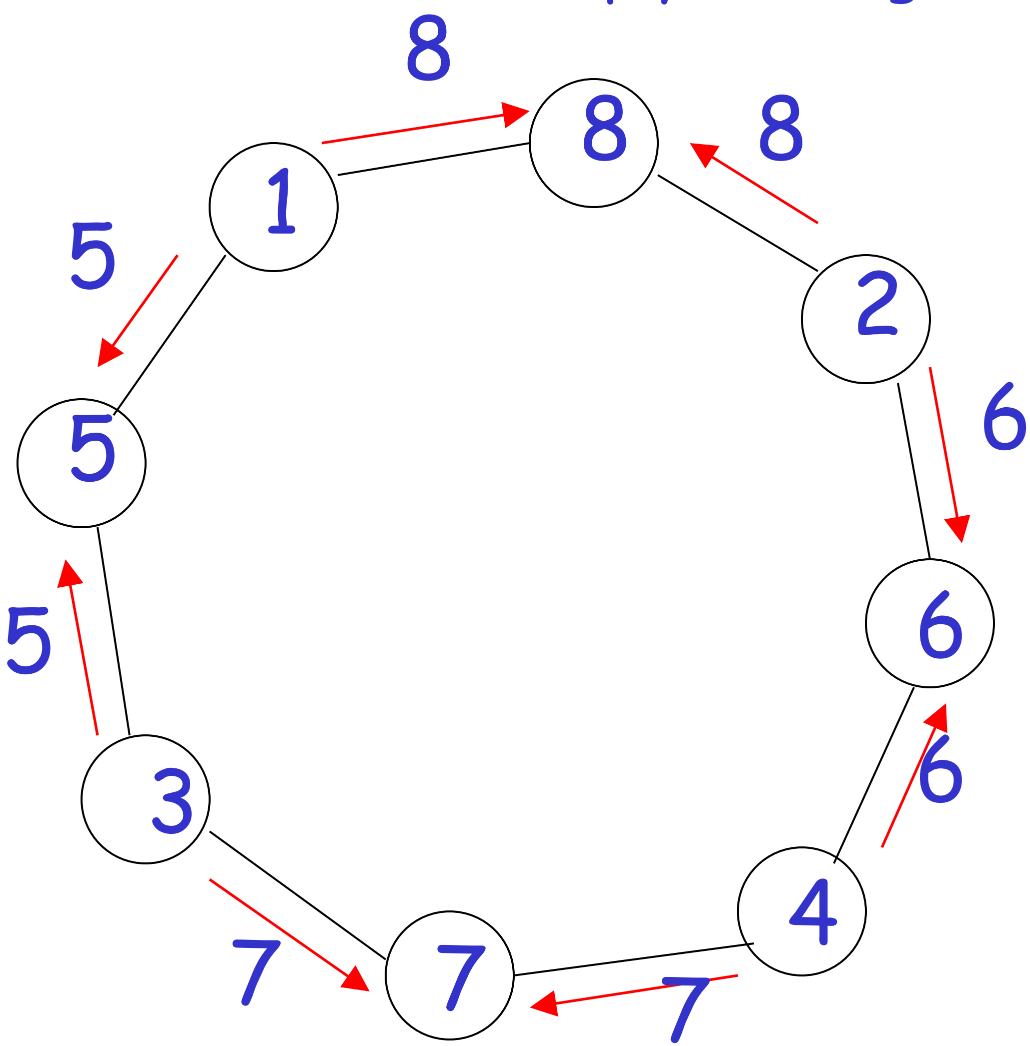
Upon receiving a message REPLY$\langle j, k\rangle$ from left (right):

```
if (my_id ≠ j) then
 send(REPLY⟨j, k⟩) to right (left);
else
  if (already received REPLY⟨j, k⟩)
   send(ELECTION⟨j, k + 1, 1⟩) to left and right;
```

Phase 0: each node sends a probing message (id, 0, 1) to its 2^0=1-neighborhood, i.e., to its left and its right

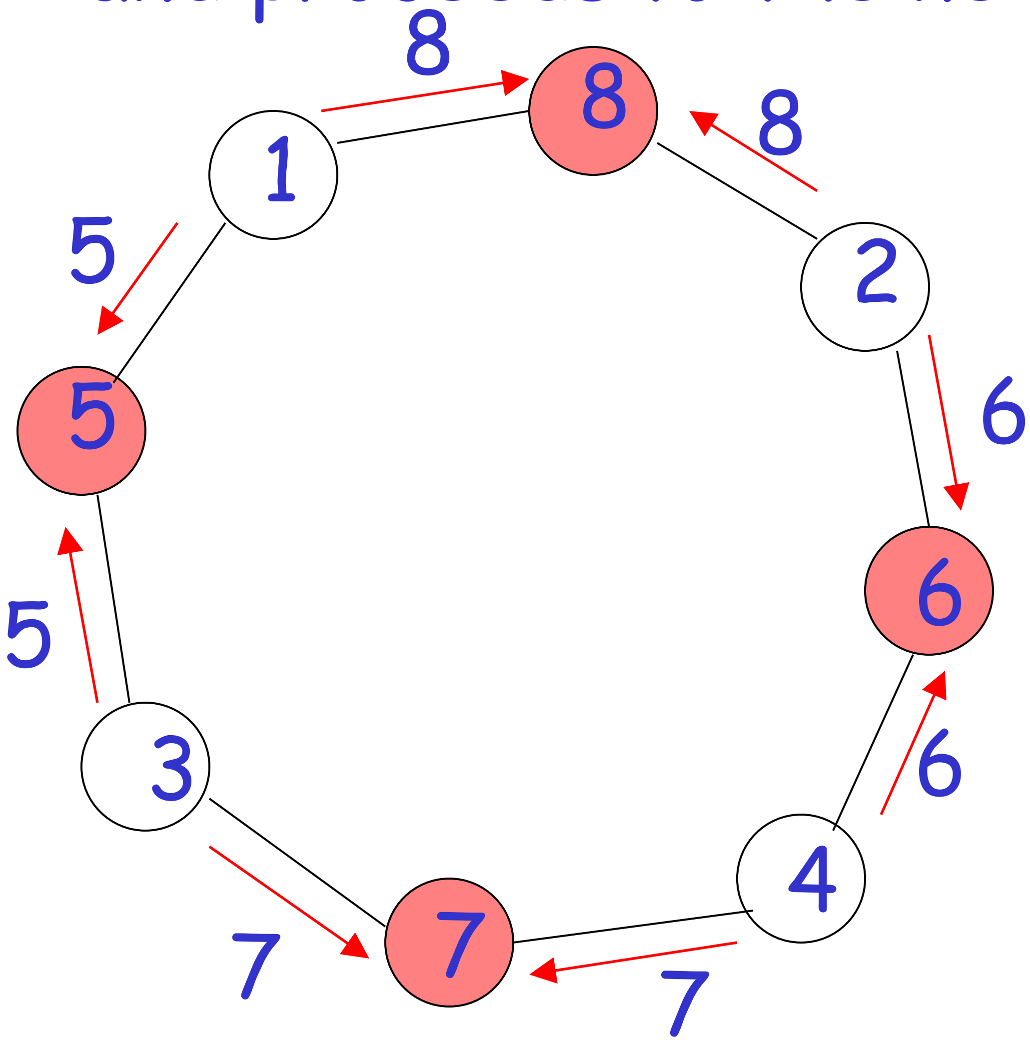


Phase 0: each node receives a probing message (id, 0, 1) from its left and its right, and so it realizes it is the last node of the neighborhood (since $2^0=1$); if the received id is greater than its own id, it sends back a reply message

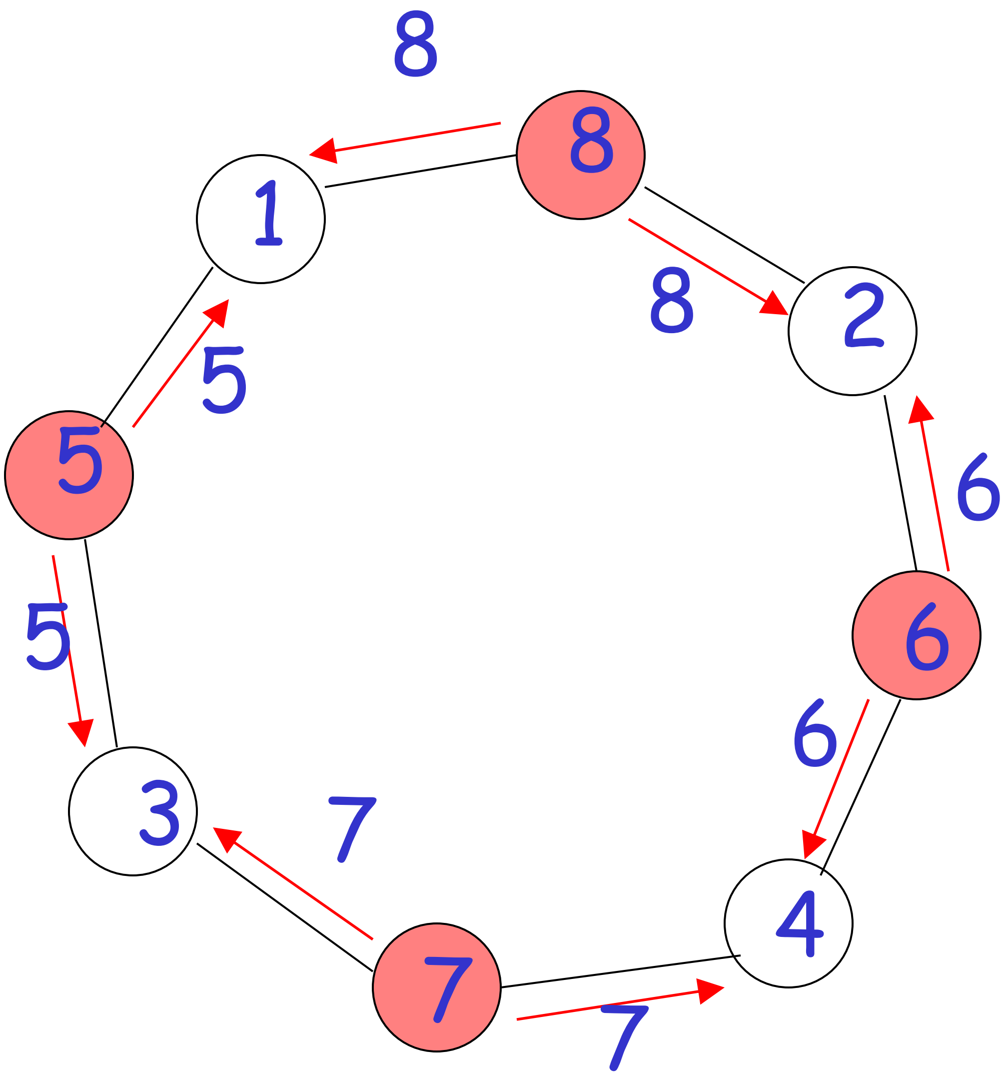


If: a node receives both replies

Then: it becomes a temporary leader and proceeds to the next phase

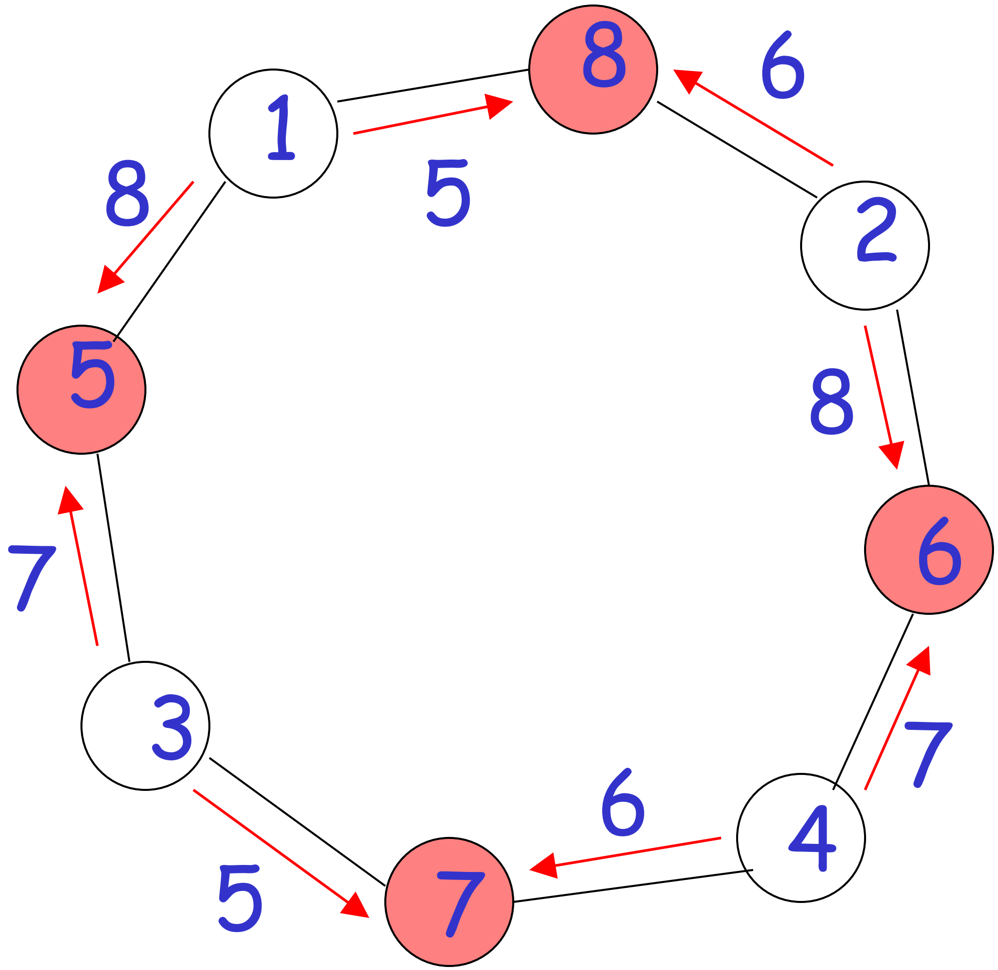


Phase 1: send a probing message (id,1,1) to left and right nodes in the 2^1-neighborhood



If: received id > my own id

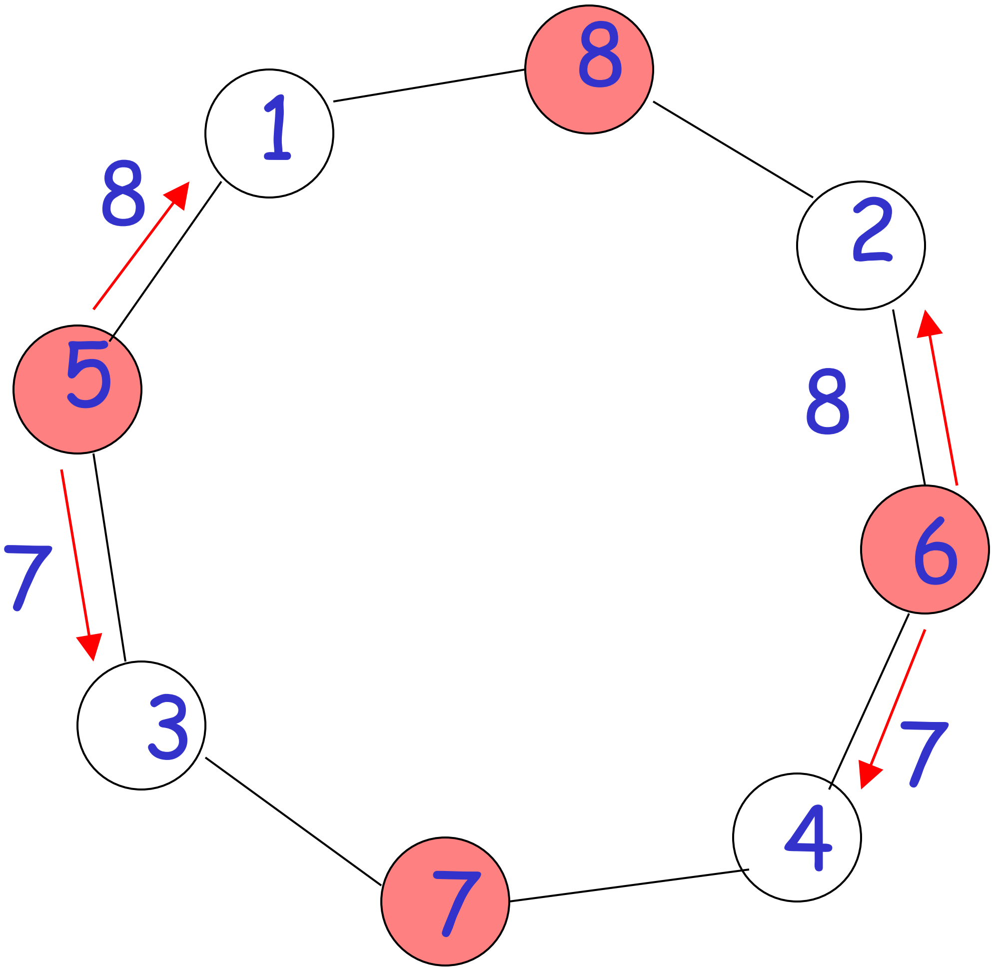
Then: forward the probing message (id,1,2)



At second step: since step counter=2, if a node receive a probing message, it realizes it is on the boundary of the 2-neighborood

If: received id > my own id

Then: send a reply message

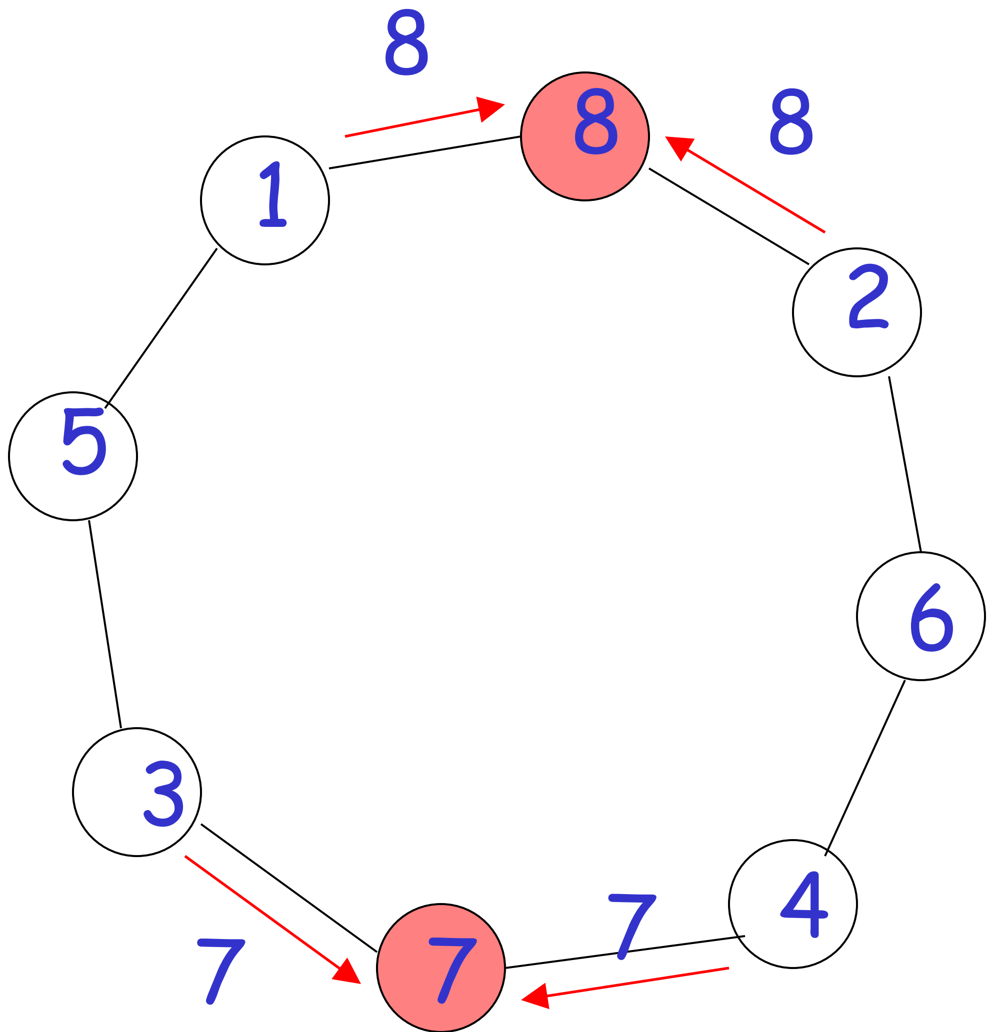


If: a node receives a reply message with another id

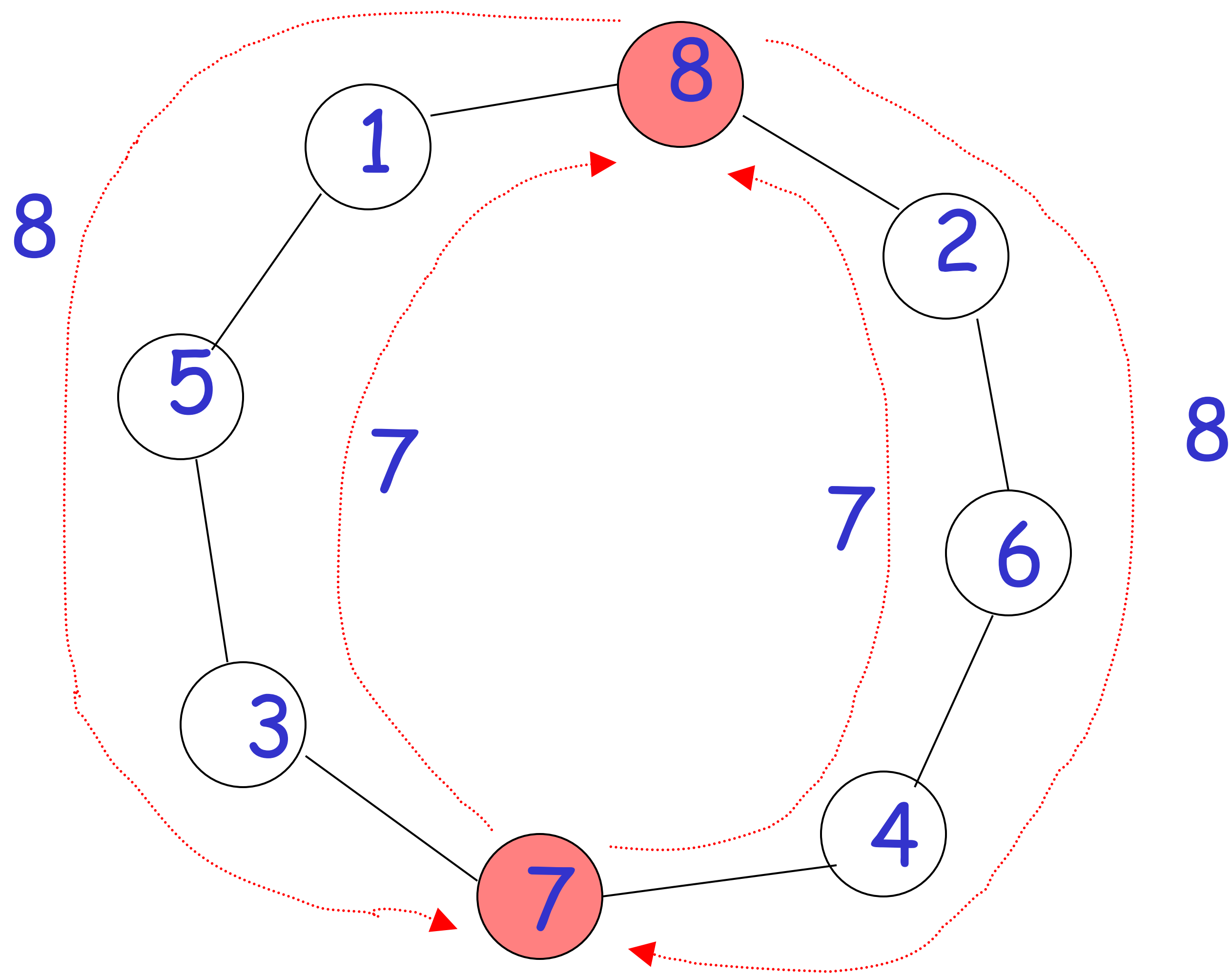
Then: forward it

If: a node receives both replies

Then: it proceed to the next phase



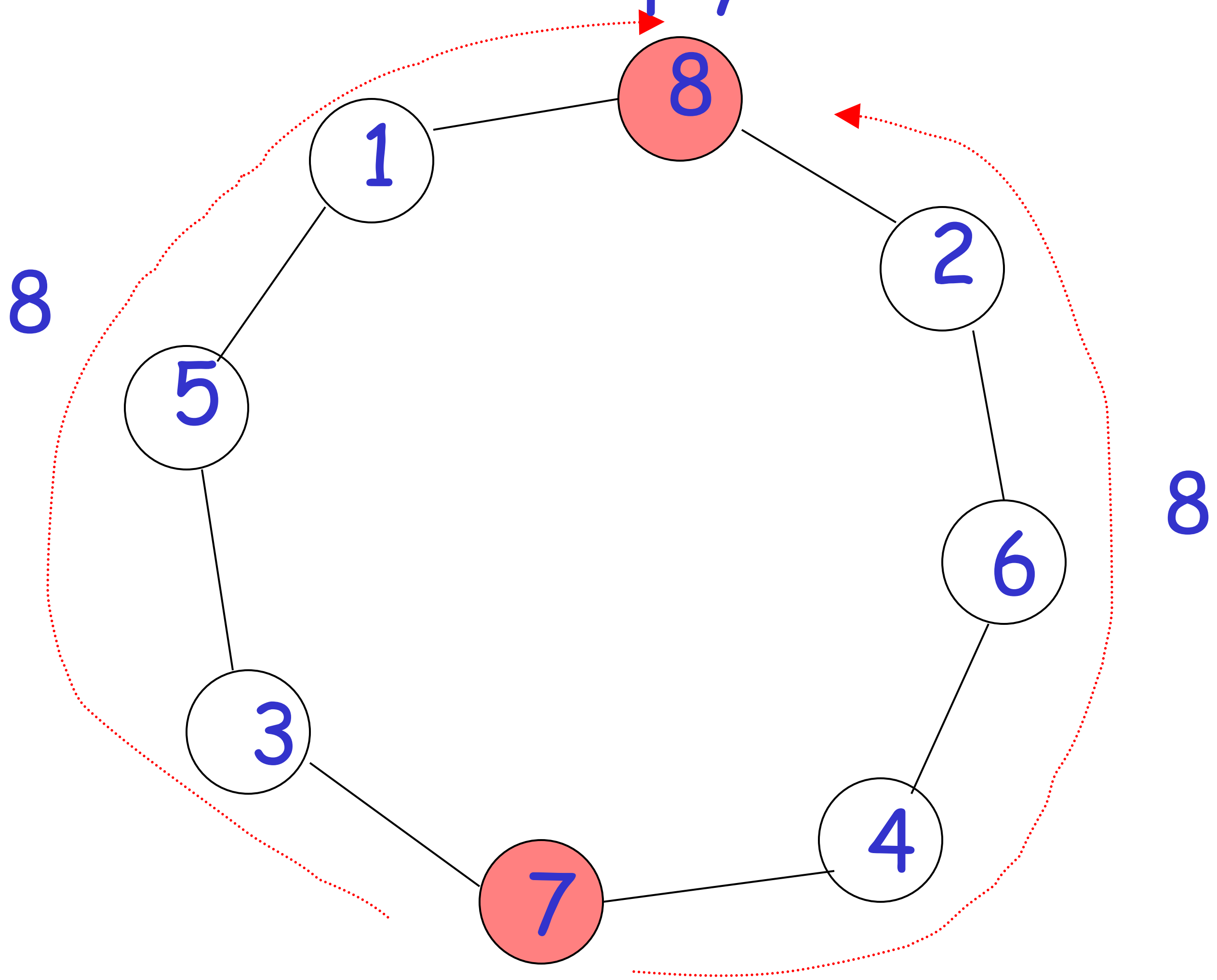
Phase 2: send id to the 2^2=4-neighborhood



At the 2^2 step:

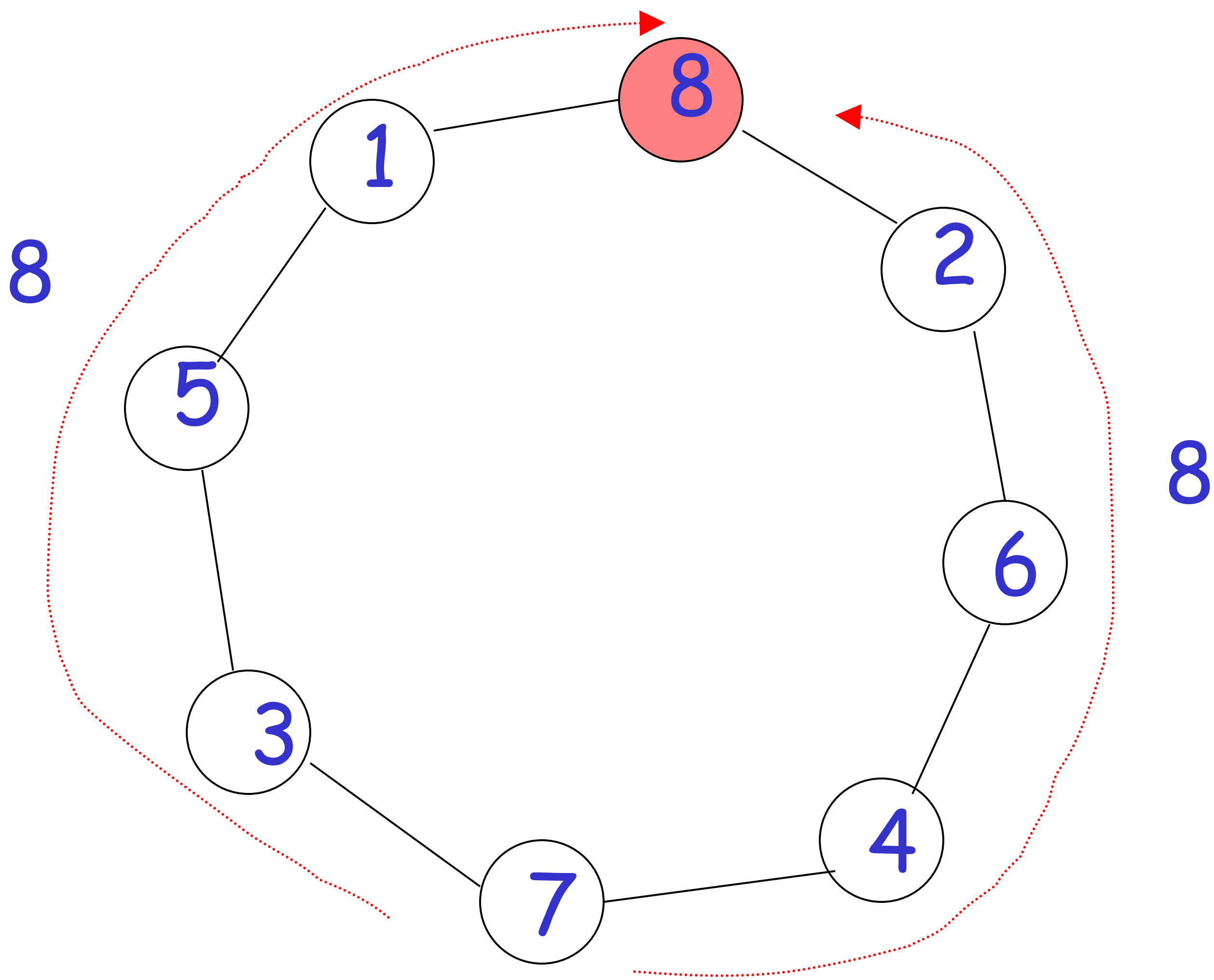
If: received id > current id

Then: send a reply



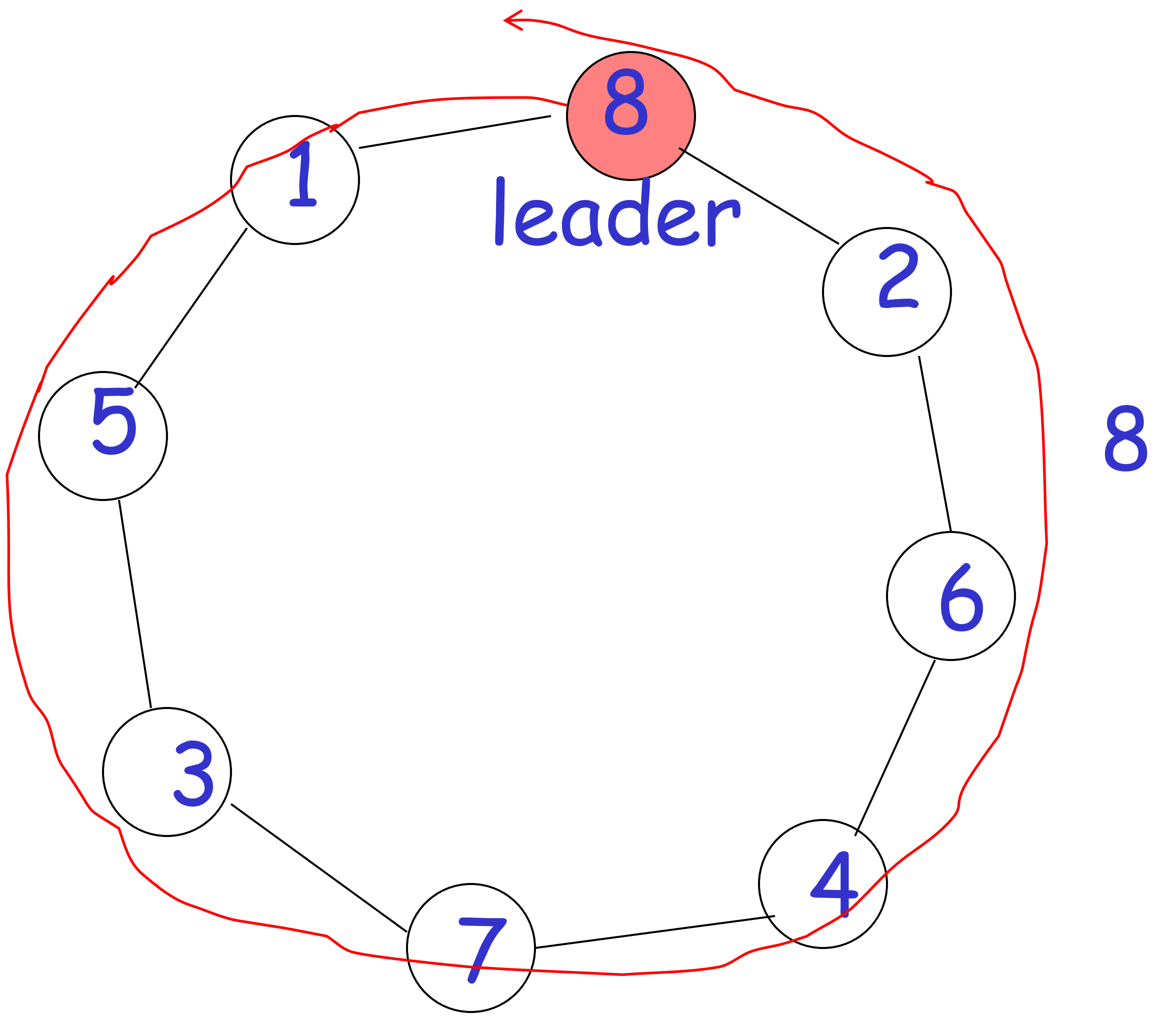
If: a node receives both replies

Then: it becomes temporary leader



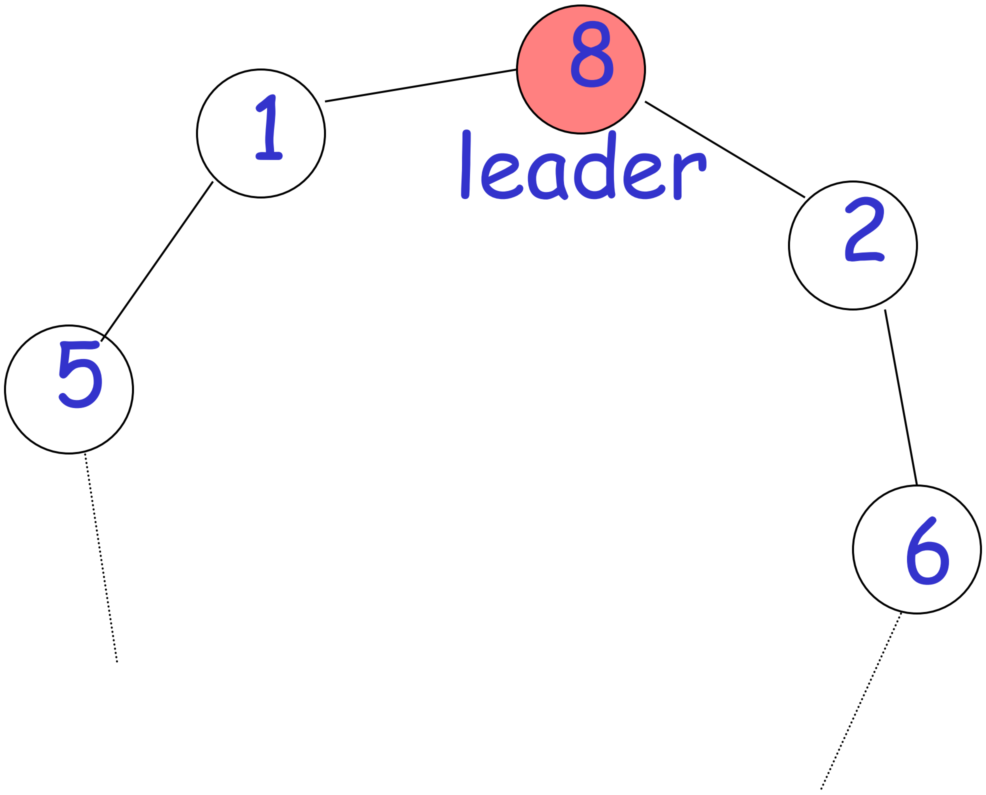
Phase 3: send id to 2^3=8-neighborhood

⇒ The node with id 8 will receive its own probe message, and then becomes the leader!



In general:

n nodes ⇨ $\Theta(\log n)$ phases



Analysis of HS algorithm

Correctness: Similar to CR algorithm.

Message Complexity:

- Each msg belongs to a particular phase and is initiated by a particular proc.
- Probe distance in phase i is 2^i
- Number of msgs initiated by a processor in phase i is at most $4 \cdot 2^i$ (probes and replies in both directions)

Message complexity

Max # messages per each node trying to become temporary leader	Max # nodes trying to become temporary leader
Phase 0: 4	n
Phase 1: 8	$n/2$
…	
Phase i: 2^{i+2}	$n/2^{i}$
…	
Phase log n: $2^{\log n+2}$	$n/2^{\log n}$

Message complexity

Max # messages per leader × Max # current leaders

Phase 0: $4 \times n = 4n$

Phase 1: $8 \times n/2 = 4n$

...

Phase i: $2^{i+2} \times n/2^{i} = 4n$

...

Phase log n: $2^{\log n+2} \times n/2^{\log n} = 4n$

Total messages: $O(n \cdot \log n)$

Can we do better?

- The $O(n \log n)$ algorithm is more complicated than the $O(n^2)$ algorithm but uses fewer messages in the worst case.
- It works in both the synchronous and the asynchronous case (and no synchronized start is required)
- Can we reduce the number of messages even more? Not in the asynchronous model:

Thr: Any asynchronous uniform LE algorithm on a ring requires $\Omega(n \log n)$ messages.

Homework:

1. What about a best case for HS?
2. Can you see an instance of HS which will use $\Theta(n \log n)$ messages?
3. What about a variant of HS in which probing messages are sent only along one direction (for instance, on the left side)?

A Θ(n)-messages Synchronous Algorithm

Requirements: n must be known (i.e., it is non-uniform), and all the processors must start together at the very beginning (this assumption could be easily relaxed)

Reminder: At each round each processor, in order:

- Reads the incoming messages buffer;
- Makes some internal computations;
- Sends messages which will be read in the next round.

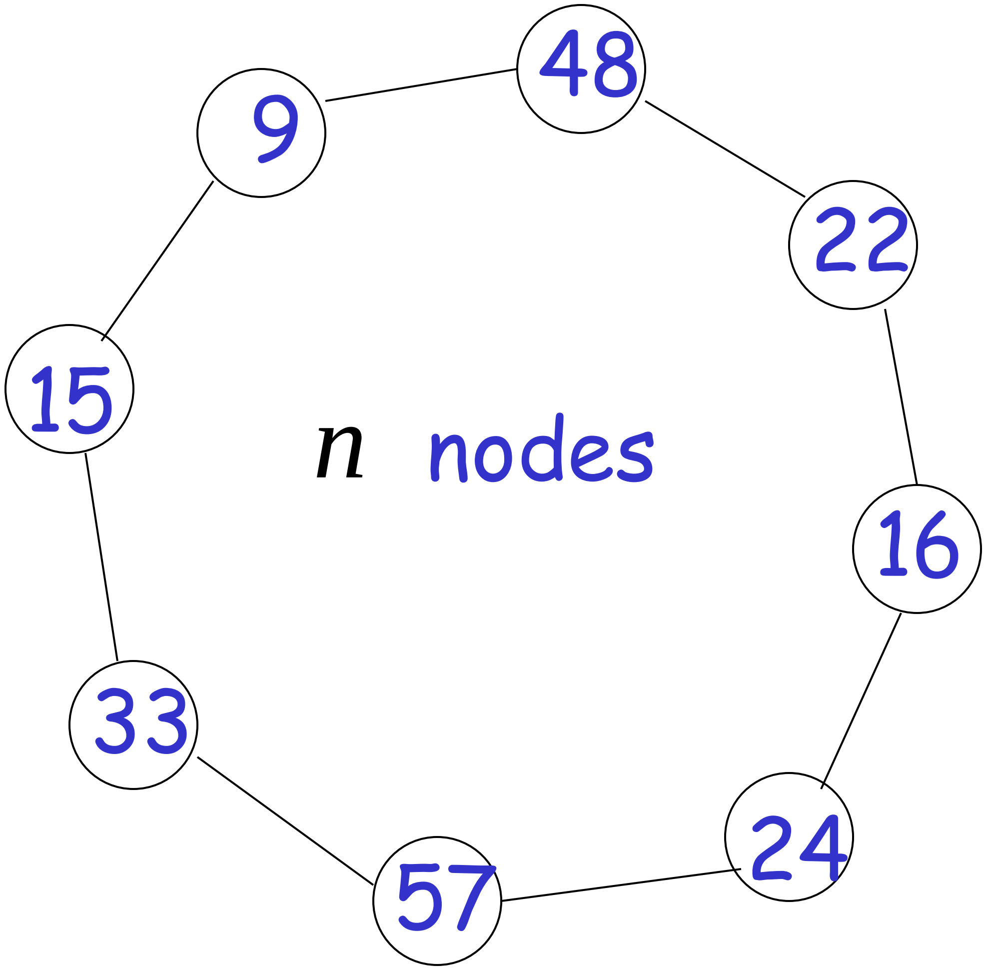
Rounds are grouped in **phases**: each phase consists of n rounds:

If in phase k=0,1,,... there is a node with id k

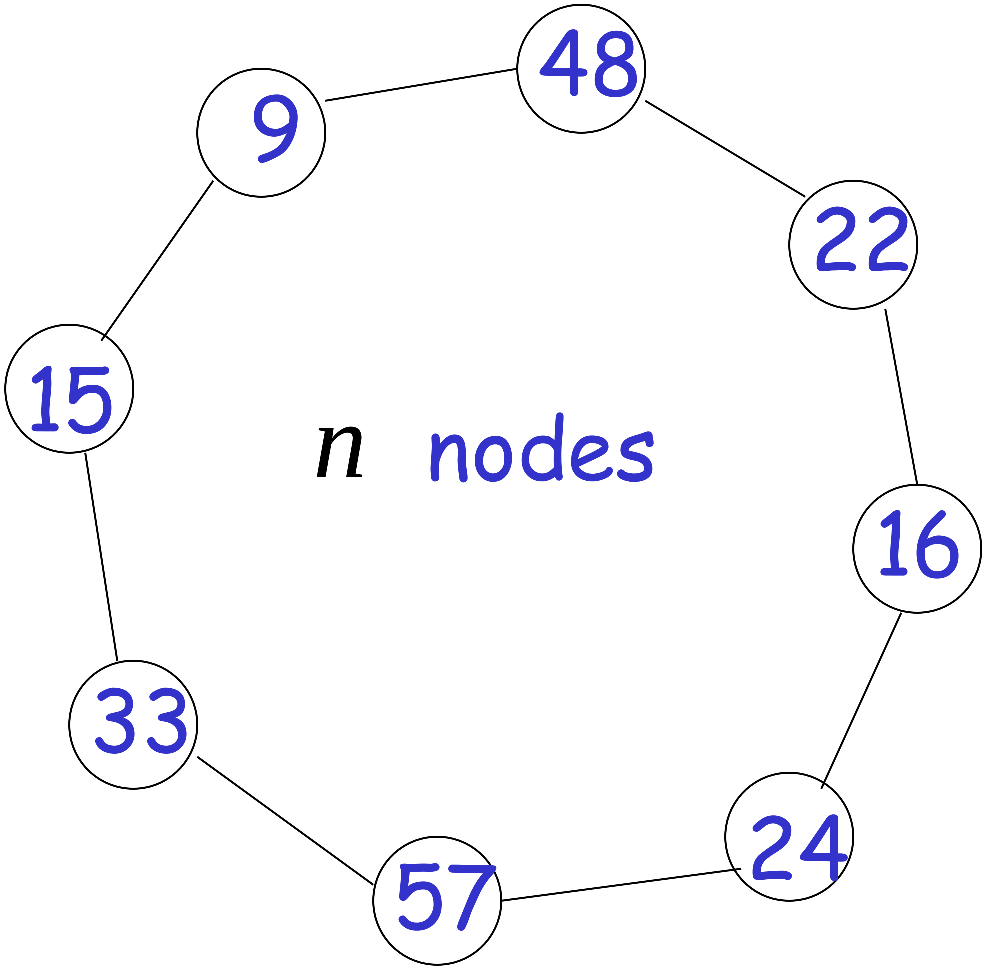
- it elects itself as the leader;
- it notifies all the other nodes it became the leader;
- the algorithm terminates.

Remark: The node with smallest id is elected leader

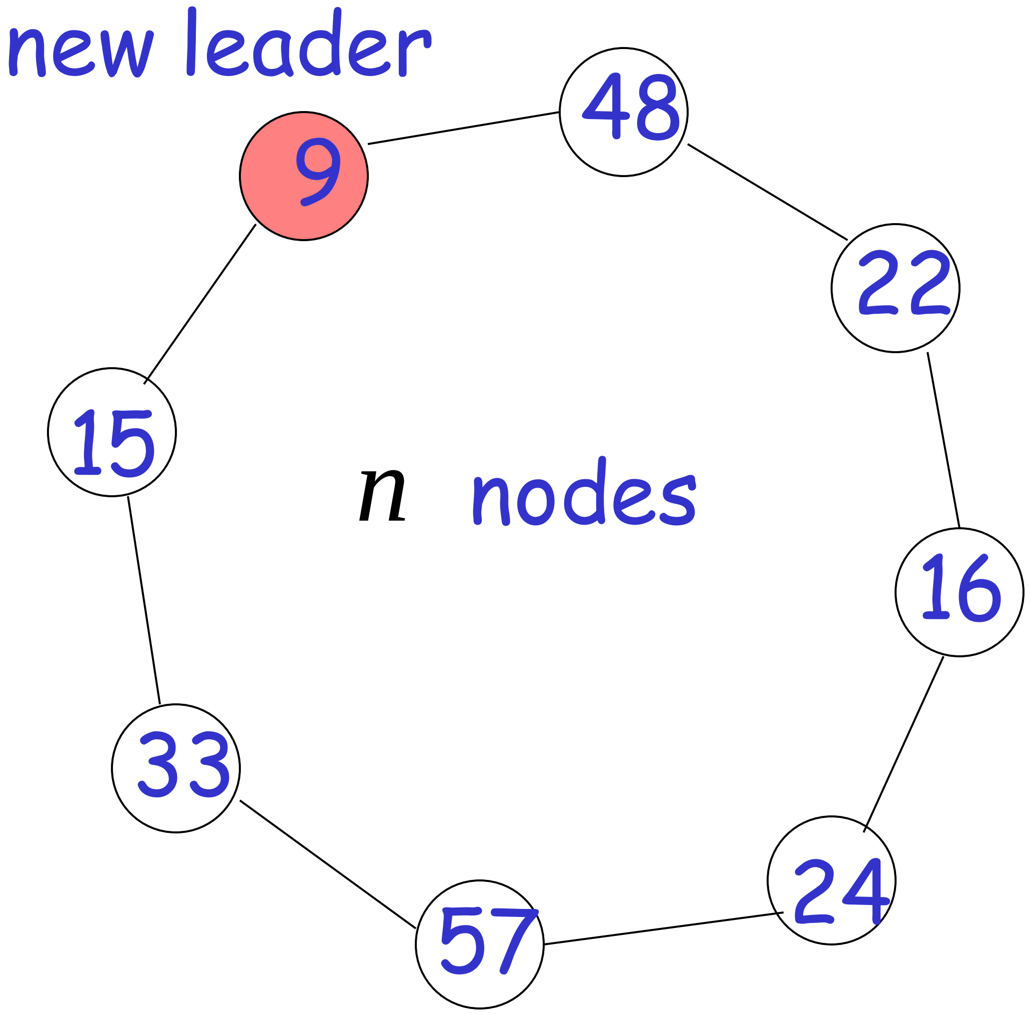
Phase 0 (n rounds): no message sent



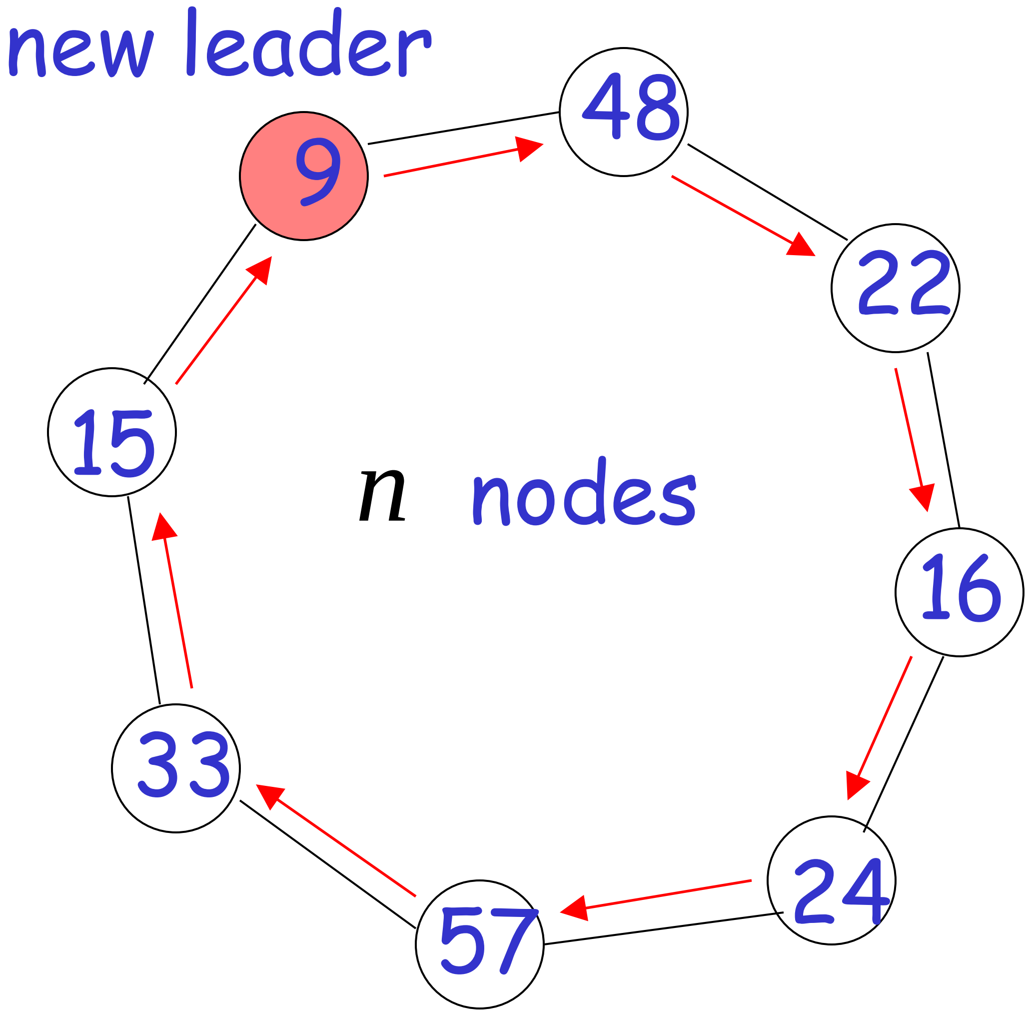
Phase 1 (n rounds): no message sent



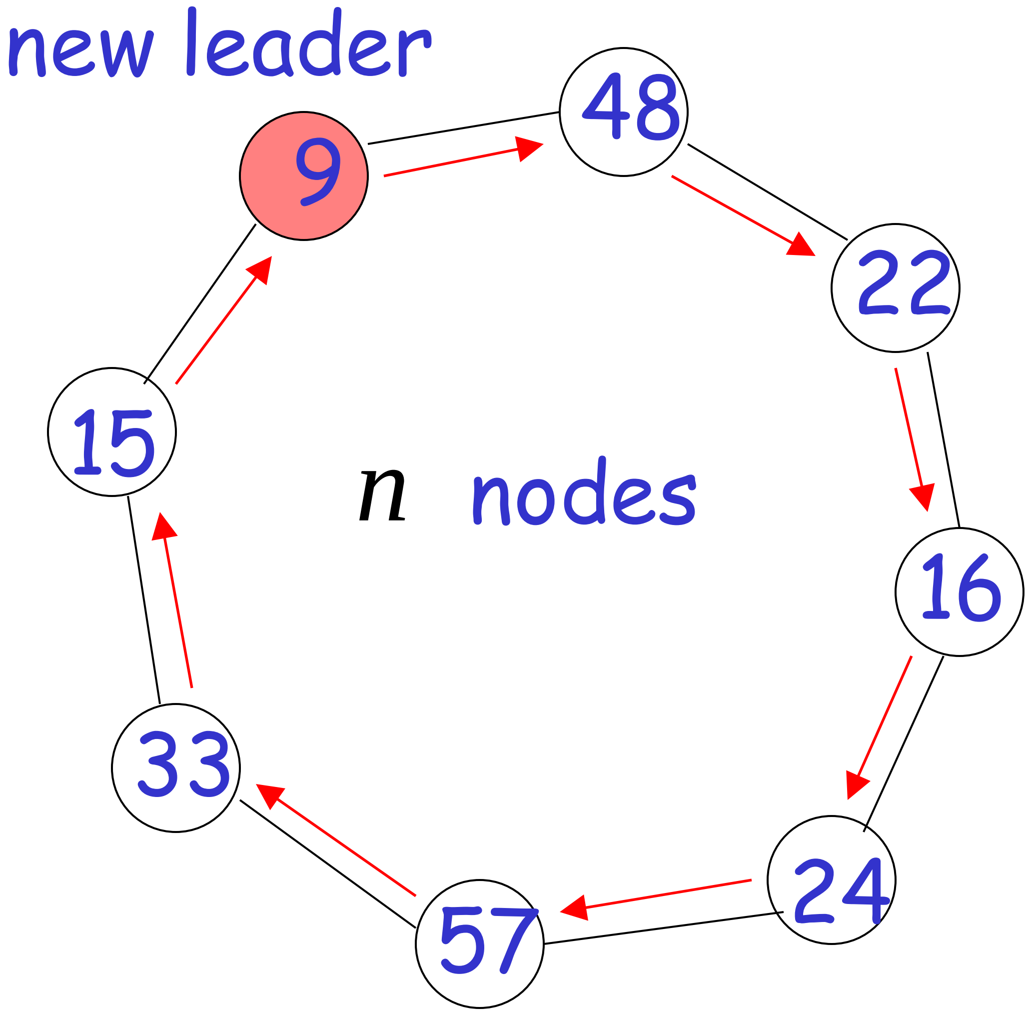
... Phase 9



Phase 9 (n rounds): n messages sent

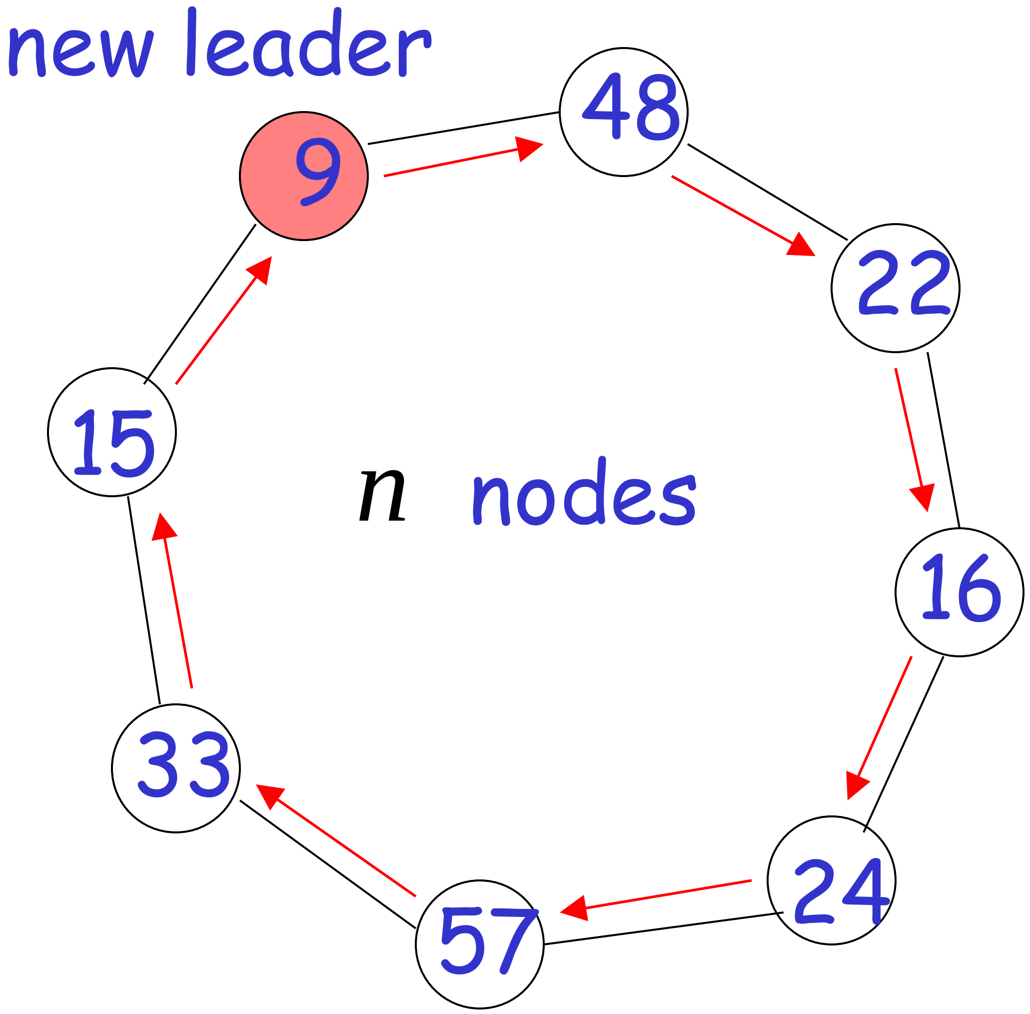


Phase 9 (n rounds): n messages sent



Algorithm Terminates

Phase 9 (n rounds): n messages sent



Total number of messages: n

Algorithm analysis

Correctness: Easy to see ☺

Message complexity: $\Theta(n)$, which can be shown to be optimal ☺

Time complexity (# rounds): $\Theta(n \cdot m)$, where m is the smallest id in the ring $\Rightarrow$ not bounded by any function of n $\Rightarrow$ it is not strongly polynomial in n. Notice however that it is commonly assumed that $m=O(n^k)$, $k=O(1)$

Other disadvantages:

- Requires synchronous start (not really!)
- Requires knowing n (non-uniform)

Another Θ(n)-messages Synchronous Algorithm: the slow-fast algorithm

Works in a weaker model than the previous synchronous algorithm:

- uniform (does not rely on knowing n)
- processors need not start at the same round; a processor either wakes up spontaneously or when it first gets a message
- IDEA: messages travel at different "speeds" (the leader's one is the fastest)

Reminder: At each round each processor, in order:

- Reads the incoming messages buffer;
- Makes some internal computations;
- Sends messages which will be read in the next round.

Another Θ(n)-messages Synchronous Algorithm: the slow-fast algorithm

- A processor that wakes up spontaneously is **active**; sends its id in a **fast** message (one edge per round) in a clockwise direction
- A processor that wakes up when receiving a msg is **relay**; it does not enter ever in the competition to become leader
- A processor only forwards a message whose id is smaller than any other competing id it has seen so far (this is different from CR algorithm)
- A fast message carrying id m that reaches an active processor becomes **slow**: it starts traveling at one edge every 2^m rounds (i.e., a processor that receives it at round r, will forward it at round $r+2^m$)
- If a processor gets its own id back, it elects itself as leader

Algorithm analysis

Correctness: convince yourself that the active processor with smallest id is elected.

Message complexity: Winner's msg is the fastest. While it traverses the ring, other messages are slower, so they are overtaken and stopped before too many messages are sent.

Message Complexity

A message will contain 2 fields: (id, 0/1 (slow/fast))

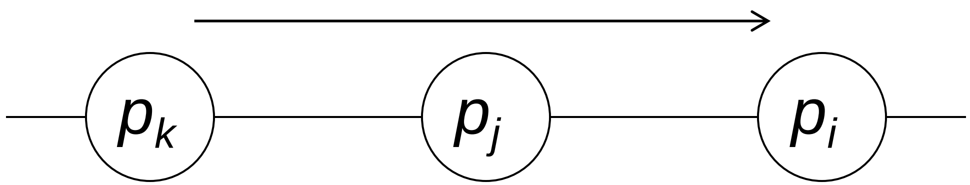
Divide msgs into four types:

1. fast msgs
2. slow msgs sent while the leader's msg is fast
3. slow msgs sent while the leader's msg is slow
4. slow msgs sent while the leader is sleeping

Next, count the number of each type of msg.

Number of Type 1 Messages
(fast messages)

Show that no processor forwards more than one fast msg (by contradiction):



Suppose p_i forwards p_j's fast msg and p_k's fast msg. But when p_k's fast msg arrives at p_j:

1. either p_j has already sent its fast msg, so p_k's msg becomes slow (contradiction)
2. p_j has not already sent its fast msg, so it never will (contradiction) since it is a relay

Number of type 1 msgs is O(n).

Number of Type 2 Messages (slow msgs sent while leader's msg is fast)

Leader's msg is fast for at most n rounds

by then it would have returned to leader

Slow msg i is forwarded $n/2^i$ times in n rounds

Max. number of msgs is when ids are as small as possible (0 to n-1 and leader is 0)

Number of type 2 msgs is at most

$$\sum_{i=1}^{n-1} n/2^i \leq n$$

Number of Type 3 Messages (slow msgs sent while leader's msg is slow)

Maximum number of rounds during which leader's msg is slow is $n \cdot 2^L$ (L is leader's id).

No msgs are sent once leader's msg has returned to leader

Slow msg i is forwarded $n \cdot 2^L / 2^i$ times during $n \cdot 2^L$ rounds.

Worst case is when ids are L to L+n-1 (independently of L, and so in particular, when L=0)

Number of type 3 msgs is at most

$$\sum_{i=L}^{L+n-1} n \cdot 2^L / 2^i \le 2n$$

Number of Type 4 Messages
(slow messages sent while leader is sleeping)

Claim: Leader sleeps for at most n rounds.

Proof: Indeed, it can be shown that the leader will awake after at most $k \le n$ rounds, where k is the distance in the ring between the leader and the closest counter-clockwise active processor which woke-up at round 1 (prove by yourself by using induction)

- Slow message i is forwarded $n/2^i$ times in n rounds
- Max. number of messages is when ids are as small as possible (0 to n-1 and leader is 0)
- Number of type 4 messages is at most

$$\sum_{i=1}^{n-1} n/2^i \le n$$

Total Number of Messages

We showed that:

number of type 1 msgs is at most n

number of type 2 msgs is at most n

number of type 3 msgs is at most 2n

number of type 4 msgs is at most n

$\Rightarrow$ total number of msgs is at most $5n=O(n)$, and of course is at least n, and so the message complexity is $\Theta(n)$

Time Complexity

Running time is $O(n \cdot 2^m)$, where m is the smallest id. Even worse than previous algorithm, which was $O(n \cdot m)$. This algorithm is polynomial in n only if we assume that the smallest identifier is $O(\log n)$ (which is realistic, though)

$\Rightarrow$ The advantage of having a linear number of messages is paid by both the synchronous algorithms with a number of rounds which depends on the minimum id ☹

Summary of LE algorithms on rings

- Anonymous rings: no any algorithm
- Non-anonymous asynchronous rings:
  - $O(n^2)$ algorithm (unidirectional rings)
  - $O(n \log n)$ messages (optimal, bidirectional rings)
- Non-anonymous synchronous rings:
  - $\Theta(n)$ messages (optimal), $O(nm)$ rounds (non-uniform, all processors must start at round 1)
  - $\Theta(n)$ messages (optimal), $O(n2^m)$ rounds (uniform)

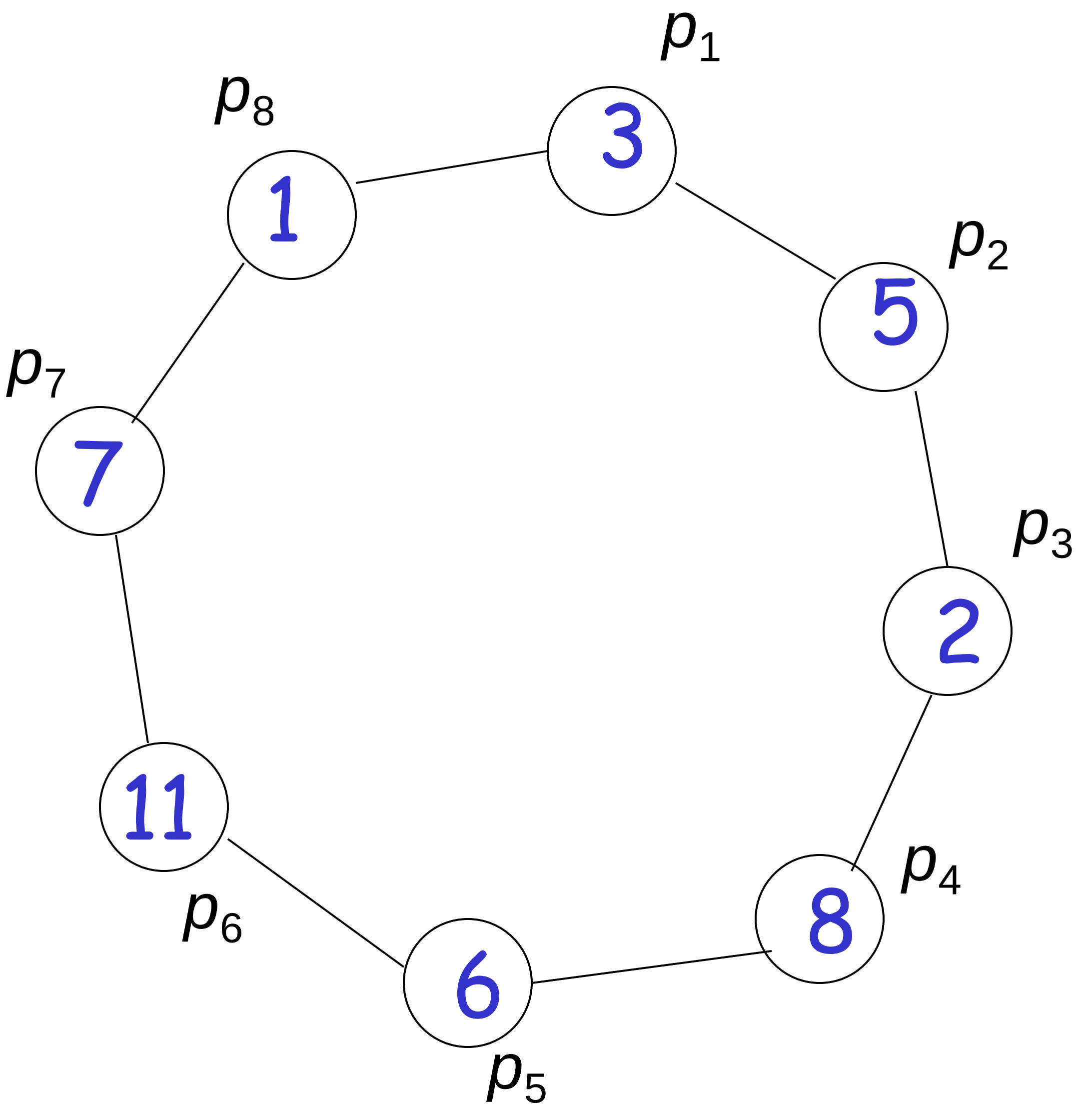
LE algorithms on general topologies

INPUT: a MPS G=(V,E) with |V|=n and |E|=m

- Anonymous: no any algorithm (of course…)
- Non-anonymous asynchronous systems:
  - $O(m+n \log n)$ messages
- Non-anonymous synchronous systems:
  - $O(m+n \log n)$ messages, $O(n \log n)$ rounds

- Homework: think to complete graphs…

Homework: Write the pseudo-code and execute the slow-fast algorithm on the following ring, assuming that p_1, p_5, p_8 will awake at round 1, and p_3 will awake at round 2.



Pseudocode

```
TYPE MSG{
    int ID
    boolean SPEED // 0=SLOW; 1=FAST}

PROGRAM MAIN{//Start at any round
    either spontaneously or after
    receiving a message
  STATE:=Non_Leader
  SMALLER_ID:=+∞
  R:= current round //taken from the
    universal clock
  IF(IN_BUFFER=Empty){
     SMALLER_ID:=MY_ID
     MSG.ID:=MY_ID
     MSG.SPEED:=1
     SEND(MSG)
     REPEAT(ACTIVE_CASE)
  } ELSE REPEAT(RELAY_CASE)
}
```

```
PROCEDURE ACTIVE_CASE{//This is repeated in any round
following the waking-up round
   R:= current round
   IF(IN_BUFFER=Non-Empty){
      RECEIVE(MSG)  //This makes the IN_BUFFER empty
      IF(MSG.ID=MY_ID){
          STATE:=Leader
          EXIT}
      IF(MSG.ID < SMALLER_ID){
          SMALLER_ID:=MSG.ID
          TIMEOUT:=R+(2^MSG.ID)-1
          MSG.SPEED:=0;
          OUT_BUFFER:=MSG
          }}
   IF(R=TIMEOUT) SEND(OUT_BUFFER)
}

PROCEDURE RELAY_CASE{//This is repeated in any round
since the waking-up round
   R:= current round
   IF(IN_BUFFER=Non-Empty){
     RECEIVE(MSG) //This makes the IN_BUFFER empty
     IF(MSG.ID < SMALLER_ID){
          SMALLER_ID:=MSG.ID
          OUT_BUFFER:=MSG
          IF(MSG.SPEED=1) TIMEOUT:=R
          ELSE TIMEOUT:=R+(2^MSG.ID)-1}}
   IF(R=TIMEOUT) SEND(OUT_BUFFER)
}
```