Laurea Magistrale in Informatica
Formal Methods (2015-2016)

Midterm Exam - April 22nd, 2016

Some solutions

1. Given the set $A = \{a, b, c, d, e\}$, consider two reduction relations $\longrightarrow$ and $\longrightarrow_1$ defined on A as follows:

$$\begin{array}{lcl} a & \longrightarrow & b \\ a & \longrightarrow & c \\ b & \longrightarrow & c \\ b & \longrightarrow & d \\ c & \longrightarrow & a \\ c & \longrightarrow & d \\ e & \longrightarrow & e \end{array} \qquad\qquad \begin{array}{lcl} a & \longrightarrow_1 & b \\ a & \longrightarrow_1 & d \\ b & \longrightarrow_1 & c \\ b & \longrightarrow_1 & e \\ c & \longrightarrow_1 & a \end{array}$$

Show whether $\longrightarrow$ and $\longrightarrow_1$ are weakly normalizing, noetherian, locally confluent, confluent and UN (uniqueness of normal forms). Justify your answers.

Solution

The reduction relation $\longrightarrow$ is:

- not weakly normalizing, e.g. e does not have a normal form;
- not noetherian, as there exist infinite derivations, such as loops starting from a, b, c and e;
- locally confluent, as all local confluence peaks converge: $b \longleftarrow a \longrightarrow c$ implies $b \longrightarrow c$; $c \longleftarrow b \longrightarrow d$ implies $c \longrightarrow d$; $a \longleftarrow c \longrightarrow d$ implies $a \stackrel{*}{\longrightarrow} d$;
- confluent, as all confluence peaks converge: assuming that $a/b/c$ denotes a, b or c, we have $a/b/c \stackrel{*}{\longleftarrow} a \stackrel{*}{\longrightarrow} d$ implies $a/b/c \stackrel{*}{\longrightarrow} d$ and similarly for the confluence peaks starting from b and c;
- UN, as it derives from the confluence property, but it can also be shown that all elements have at most one normal form, because a, b, c and d have the unique normal form d, while e does not have a normal form.

The reduction relation $\longrightarrow_1$ is:

- weakly normalizing, as there exists at least a normal form (either d or e) for all elements;
- not noetherian, as there exist infinite derivations, such as loops starting from a, b and c;
- locally confluent, as all local confluence peaks converge: $b \; {}_1\!\!\longleftarrow a \longrightarrow_1 d$ implies $b \stackrel{*}{\longrightarrow}_1 d$ and $c \; {}_1\!\!\longleftarrow b \longrightarrow_1 e$ implies $c \stackrel{*}{\longrightarrow}_1 e$;
- not confluent, e.g. take the confluence peak $d \; {}_1\!\!\stackrel{*}{\longleftarrow} a \stackrel{*}{\longrightarrow}_1 e$ where d and e are distinct normal forms;
- not UN, as there exist two distinct normal forms d and e for elements a, b and c.

2. Given the signature $\Sigma = \{a, f, g, h\}$ and variables $x, y, z, w \in V$, compute the (idempotent) most general unifier (if it exists) for each pair of terms:

i) $t_1 = g(h(x), g(x, y))$ and $t_2 = g(z, g(a, h(z)))$;
ii) $t_1 = f(h(x), f(x, x, y), g(y, y))$ and $t_2 = f(h(z), f(z, w, h(a)), w)$;
iii) $t_1 = f(g(h(x), h(x)), y, g(x, h(y)))$ and $t_2 = f(g(z, w), a, g(z, w))$.

Solution
i) Starting with
$$(\{g(h(x), g(x, y)) = g(z, g(a, h(z)))\}, \emptyset)$$
and applying the rule of *Decomposition* twice, we get
$$(\{h(x) = z,\ x = a,\ y = h(z)\}, \emptyset)$$
Variable Elimination on x obtains
$$(\{a/x\}\{h(x) = z,\ y = h(z)\}, \{x = a\})$$
which, by applying the binding for x to the system of equations still to be solved, becomes
$$(\{h(a) = z,\ y = h(z)\}, \{x = a\})$$
Variable Elimination on z gives
$$(\{h(a)/z\}\{y = h(z)\}, \{x = a,\ z = h(a)\})$$
which is equal to
$$(\{y = h(h(a))\}, \{x = a,\ z = h(a)\})$$
Finally we apply *Variable Elimination* on y, thus deriving
$$(\{h(h(a))/y\}\emptyset = \emptyset, \{x = a,\ z = h(a),\ y = h(h(a))\})$$
The set of equations on the left is empty and the one on the right is a system in solved form representing the mgu $\sigma = \{a/x,\ h(h(a))/y,\ h(a)/z\}$ for the given terms.

ii) By applying the rule of *Decomposition* we get
$$(\{f(h(x), f(x, x, y), g(y, y)) = f(h(z), f(z, w, h(a)), w)\}, \emptyset)$$
$$(\{h(x) = h(z),\ f(x, x, y) = f(z, w, h(a)),\ g(y, y) = w\}, \emptyset)$$
$$(\{x = z,\ x = z,\ x = w,\ y = h(a),\ g(y, y) = w\}, \emptyset)$$
Using *Variable Elimination* on y, w, x and z (and also by deleting the redundant equation $x = z$ that occurs twice), we have
$$(\{h(a)/y\}\{x = z,\ x = w,\ g(y, y) = w\} = \{x = z,\ x = w,\ g(h(a), h(a)) = w\},$$
$$\{y = h(a)\})$$
$$(\{g(h(a), h(a))/w\}\{x = z,\ x = w\} = \{x = z,\ x = g(h(a), h(a))\},$$
$$\{y = h(a),\ w = g(h(a), h(a))\})$$
$$(\{g(h(a), h(a))/x\}\{x = z\} = \{g(h(a), h(a)) = z\},$$
$$\{y = h(a),\ w = g(h(a), h(a)),\ x = g(h(a), h(a))\})$$
$$(\{g(h(a), h(a))/z\}\emptyset = \emptyset,$$
$$\{y = h(a),\ w = g(h(a), h(a)),\ x = g(h(a), h(a)),\ z = g(h(a), h(a))\})$$

obtaining the mgu $\sigma = \{g(h(a),h(a))/x,\ h(a)/y,\ g(h(a),h(a))/z,\ g(h(a),h(a))/w\}$.

iii) By applying the rules of *Decomposition* (more times) and *Variable Elimination* on y and w, then again *Decomposition* and *Variable Elimination* on x and z, finally we have *Failure1*:

$(\{f(g(h(x),h(x)),y,g(x,h(y)))=f(g(z,w),a,g(z,w))\},\emptyset)$

$(\{h(x)=z,\ h(x)=w,\ y=a,\ x=z,\ h(y)=w\},\emptyset)$

$(\{a/y\}\{h(x)=z,\ h(x)=w,\ x=z,\ h(y)=w\} = \{h(x)=z,\ h(x)=w,\ x=z,\ h(a)=w\},$
$\{y=a\})$

$(\{h(a)/w\}\{h(x)=z,\ h(x)=w,\ x=z\} = \{h(x)=z,\ h(x)=h(a),\ x=z\},$
$\{y=h(a),\ w=h(a)\})$

$(\{h(x)=z,\ x=a,\ x=z\},\{y=a,\ w=h(a)\})$

$(\{a/x\}\{h(x)=z,\ x=z\} = \{h(a)=z,\ a=z\},\{y=a,\ w=h(a),\ x=a\})$

$(\{a/z\}\{h(a)=z\} = \{h(a)=a\},\{y=a,\ w=h(a),\ x=a,\ z=a\})$

$Failure$

because $h\neq a$. Hence, the two given terms are not syntactically unifiable.

3. Consider the following trs R on the signature $\Sigma = \{a,b,f,g,h\}$:

$$\begin{array}{rcl@{\qquad\qquad}rcl} f(a,x) & \to & a & g(a,x) & \to & x \\ f(x,a) & \to & x & g(b,x) & \to & h(x) \\ f(h(x),h(y)) & \to & f(x,y) & g(h(x),y) & \to & h(g(x,y)) \end{array}$$

i) Give a *reduction ordering on terms* such that R is terminating with respect to such an ordering. Show the formal steps that justify your answer.
ii) Reduce the term $t = g(h(h(a)),f(h(b),h(a)))$ to normal form in R, by applying all possible reductions from t in R using an *innermost* strategy in case of more redexes (i.e. apply the reduction on the redex at the deepest position). Show the rule applied, the position of the redex and the match for each reduction step.
iii) Given the term $s = f(g(a,h(z)),g(b,h(z)))$ where $z\in V$, compute its normal form in R.

Solution
i) We show that R is terminating with respect to an rpo $\succ_{rpo}$ and derive the precedences on Σ. By Lankford Theorem we prove that $l \succ_{rpo} r$ for each rule $l\to r$ in R.

1. $f(a,x) \succ_{rpo} a$ by subterm property.

2. $f(x,a) \succ_{rpo} x$ by subterm property or clause (iv) of gen. rpo.

3. $f(h(x),h(y)) \succ_{rpo} f(x,y)$ iff (by rpo def. with $f{=}f$) $\{h(x),h(y)\} \succ\!\succ_{rpo} \{x,y\}$ iff (multiset ordering def.) $h(x) \succ_{rpo} x$ and $h(y) \succ_{rpo} y$, which are both verified by the subterm property or clause (iv) of gen. rpo.

4. $g(a,x) \succ_{rpo} x$ by subterm property or clause (iv) of gen. rpo.

5. $g(b,x) \succ_{rpo} h(x)$ iff (by assuming the precedence $g > h$ and applying the rpo def.) $g(b,x) \succ_{rpo} x$, which is verified by the subterm property or clause (iv) of gen. rpo.

6. $g(h(x),y) \succ_{rpo} h(g(x,y))$ because, having assumed $g > h$, we prove $g(h(x),y) \succ_{rpo} g(x,y)$ iff $\{h(x),y\} \succ\!\succ_{rpo} \{x,y\}$ iff (multiset ordering def.) $h(x) \succ_{rpo} x$, which is true by the subterm property or clause (iv) of gen. rpo.

Thus R is terminating with respect to an rpo based on $g > h$.

ii) The term $t = g(h(h(a)), f(h(b), h(a)))$ has an innermost redex in $p=2$ for rule 3 with match $\sigma = \{b/x,\ a/y\}$ (there is also an outermost redex in $p=\epsilon$ for rule 6). Thus we have $t = g(h(h(a)), f(h(b), h(a))) \ \rightarrow\ g(h(h(a)), f(b,a))$.
Here the subterm in $p=2$ is an innermost redex for rule 2 with match $\sigma = \{b/x\}$: $g(h(h(a)), f(b,a)) \ \rightarrow\ g(h(h(a)), b)$.
This term can be rewritten in $p=\epsilon$ with rule 6 and match $\sigma = \{h(a)/x,\ b/y\}$: $g(h(h(a)), b) \ \rightarrow\ h(g(h(a), b))$.
Applying a rewriting step in $p=1$ with rule 6 and match $\sigma = \{a/x,\ b/y\}$ gives: $h(g(h(a), b)) \ \rightarrow\ h(h(g(a,b)))$.
Finally, using rule 4 on $p=1.1$ with match $\sigma = \{b/x\}$, we get the term $h(h(b))$ which is in normal form in R.

iii) By applying rule 4 (with $p=1$ and $\sigma = \{h(z)/x\}$), rule 5 (with $p=2$ and $\sigma = \{h(z)/x\}$) and rule 3 (with $p=\epsilon$ and $\sigma = \{z/x, h(z)/y\}$), we have the following derivation:
$s = f(g(a, h(z)), g(b, h(z))) \rightarrow f(h(z), g(b, h(z))) \rightarrow f(h(z), h(h(z))) \rightarrow f(z, h(z))$.
The last term is the normal form of s in R.

4. Given the signature $\Sigma = \{f, g, h, k\}$, orient the following equations

$$\begin{aligned} f(g(y,x), k(y)) &= k(g(y,x)) \\ f(g(x,z), g(z,y)) &= g(z, f(x,y)) \\ h(x,y) &= f(k(x), y) \end{aligned}$$

into a TRS R. Give a *reduction ordering on terms* such that R is terminating with respect to such an ordering. Show the formal steps that justify your answer.

Solution
Let us show that the TRS R obtained by orienting all equations from left to right is terminating with respect to an rpo.

1. $f(g(y,x), k(y)) \succ_{rpo} k(g(y,x))$ if we assume $f > k$ and prove $f(g(y,x), k(y)) \succ_{rpo} g(y,x)$, which is true by the subterm property.

2. $f(g(x,z), g(z,y)) \succ_{rpo} g(z, f(x,y))$ if we assume $f > g$ and prove $\{f(g(x,z), g(z,y))\} \succ\!\succ_{rpo} \{z, f(x,y)\}$ iff (multiset ordering def.) $f(g(x,z), g(z,y)) \succ_{rpo} z$ (true by the subterm property or clause (iv) of gen. rpo) and $f(g(x,z), g(z,y)) \succ_{rpo} f(x,y)$ iff ($f{=}f$) $\{g(x,z), g(z,y)\} \succ\!\succ_{rpo} \{x,y\}$ iff $g(x,z) \succ_{rpo} x$ and $g(z,y) \succ_{rpo} y$, which are both verified by the subterm property or clause (iv) of gen. rpo.

3. $h(x,y) \succ_{rpo} f(k(x), y)$ if we assume $h > f$ and prove $\{h(x,y)\} \succ\!\succ_{rpo} \{k(x), y\}$ iff

$h(x, y) \succ_{rpo} k(x)$ and $h(x, y) \succ_{rpo} y$. The latter is true by the subterm property or clause (iv) of gen. rpo. The former is true iff (as $h > f$ and $f > k$ implies $h > k$ by transitivity) $h(x, y) \succ_{rpo} x$, which is true by the subterm property or clause (iv) of gen. rpo.

Thus the TRS R obtained by orienting all equations from left to right is terminating with respect to the rpo based on $h > f > k$ and $f > g$ (this also implies $h > g$).
Note that it is possible to orient the given equations in a different way and obtain different rewrite systems that can be proved terminating with respect to different rpo.