University of L'Aquila

Master Degree in Computer Science

Course on Formal Methods

Syntactic unification: some exercises

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We refer to the *algorithm of the solved form* by Martelli and Montanari (see the lecture notes on rewrite systems) and consider some problems of syntactic unification.

1. Given the terms $t_1 = g(x_1, f(x_1))$ and $t_2 = g(x, y)$, compute (if it exists) the mgu for t_1 and t_2.

The initial system of equations is $E = \{t_1 = t_2\}$, thus the algorithm of the solved form starts with the initial configuration $(\{g(x_1, f(x_1)) = g(x, y)\}, \emptyset)$. As the topmost function symbols of the two terms of the only equation in E are the same, we can apply the inference rule *Decomposition* and get the new system $(\{x_1 = x,\ f(x_1) = y\}, \emptyset)$.

Using the rule of *Variable Elimination* it is possible to establish the first binding between a variable and a term. Such a rule can be non-deterministically applied on either equation, as both satisfy the rule precondition (that is, in both equations at least one side is a variable and such variable does not occur in the other side of the same equation). Let us consider the equation $x_1 = x$ and establish the binding $\{x/x_1\}$ which is applied to the remaining equations, thus obtaining $(\{x/x_1\}\{f(x_1) = y\},\ \{x_1 = x\}) = (\{f(x) = y\},\ \{x_1 = x\})$.

The application of the rule of *Variable Elimination* on the remaining equation results in the final configuration $(\{\emptyset, \{x_1 = x,\ y = f(x)\}\})$, whose second component is a system of equations in solved form representing the mgu $\sigma = \{x/x_1,\ f(x)/y\}$. Indeed we have $\sigma(t_1) = \sigma(t_2) = g(x, f(x))$.

Note that an mgu is unique modulo variable renaming. By applying the rule of *Variable Elimination* on the equation $x_1 = x$, whose sides are both variables, so that the other possible binding $\{x_1/x\}$ is defined, we get another mgu for t_1 and t_2 given by $\sigma' = \{x_1/x, f(x_1)/y\}$, that is in particular a match, as its domain includes only variables of the term t_2. Note that both most general unifiers are idempotent by definition of system in solved form.

2. Given the terms $t_1 = g(f(x), g(x, y))$ and $t_2 = g(x', y')$, compute (if it exists) the mgu for t_1 and t_2.

By applying the rules of *Decomposition* and *Variable Elimination* we have:

$$
\begin{aligned}
&(\{g(f(x), g(x, y)) = g(x', y')\}, \emptyset)\\
&(\{f(x) = x',\ g(x, y) = y'\}, \emptyset)\\
&(\{f(x)/x'\}\{g(x, y) = y'\} = \{g(x, y) = y'\},\ \{x' = f(x)\})\\
&(\emptyset, \{x' = f(x),\ y' = g(x, y)\})
\end{aligned}
$$

that results in the mgu $\sigma = \{f(x)/x',\ g(x, y)/y'\}$, that is a match.

3. Given the terms $t_1 = g(x, x)$ and $t_2 = g(y, f(y))$, compute (if it exists) the mgu for t_1 and t_2.

By applying the rules of *Decomposition*, *Variable Elimination* and *Failure 2*, we have:

$$
\begin{aligned}
&(\{g(x, x) = g(y, f(y))\}, \emptyset)\\
&(\{x = y,\ x = f(y)\}, \emptyset)\\
&(\{y/x\}\{x = f(y)\} = \{y = f(y)\},\ \{x = y\})\\
&Failure
\end{aligned}
$$

because $y \in Var(f(y))$. Hence, the two terms t_1 and t_2 are not syntactically unifiable. Actually, the term t_1 requires the two arguments of g to be equal (as they are denoted by the same variable x), while the two arguments of t_2 are not equal (the second one has one more f). Those terms containing more than one occurrence of the same variable, like $g(x, x)$ and $g(y, f(y))$, are called *non-linear*. Typically, when one tries to syntactically unify them, non-linear terms have more constraints (and are thus less easily unifiable) than *linear* terms where each variable occurs only once.

4. Given the terms $t_1 = g(x, x)$ and $t_2 = g(y, a)$, compute (if it exists)

the mgu for t_1 and t_2.

By applying the rules of *Decomposition* and *Variable Elimination*, we have:

$$(\{g(x,x) = g(y,a)\}, \emptyset)$$
$$(\{x = y,\ x = a\}, \emptyset)$$
$$(\{a/x\}\{x = y\} = \{a = y\},\ \{x = a\})$$
$$(\emptyset, \{x = a,\ y = a\})$$

obtaining the mgu $\sigma = \{a/x, a/y\}$.

5. Given the terms $t_1 = f(x, f(y,z))$ and $t_2 = f(x', y')$, compute (if it exists) the mgu for t_1 and t_2.

By applying the rules of *Decomposition* and *Variable Elimination*, we have:

$$(\{f(x, f(y,z)) = f(x', y')\}, \emptyset)$$
$$(\{x = x',\ f(y,z) = y'\}, \emptyset)$$
$$(\{x/x'\}\{f(y,z) = y'\} = \{f(y,z) = y'\},\ \{x' = x\})$$
$$(\emptyset, \{x' = x,\ y' = f(y,z)\})$$

resulting in the mgu $\sigma = \{x/x',\ f(y,z)/y'\}$, that is in particular a match. Modulo variable renaming, another mgu is $\sigma' = \{x'/x,\ f(y,z)/y'\}$.