University of L'Aquila

Master Degree in Computer Science

Course on Formal Methods

Termination of Rewrite Systems: some exercises

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(I) *Termination problems where the reduction ordering $\succ$ on terms is given.*

Let us consider the two situations in which either a rewrite system (trs) R or an equational theory E is given.

(I.1) *Check that a trs R is terminating with respect to a reduction ordering $\succ$.*
Let the following trs R be defined on the signature $\Sigma = \{a, f, g, h\}$:

$$\begin{array}{rcl} f(a, x) & \rightarrow & a \\ f(h(x), y) & \rightarrow & g(y, f(x, y)) \\ g(h(a), x) & \rightarrow & h(x) \end{array}$$

Check that R is terminating with respect to an rpo $\succ_{rpo}$ based on the precedences $f > g > h$.

By Lankford Theorem, in order to derive that R is terminating with respect to a reduction ordering $\succ$, it is sufficient that for each rule $l \rightarrow r$ in R we have $l \succ r$. In this example the given reduction ordering is an rpo and we know that each rpo is a simplification ordering, thus the subterm property holds.

The first rule $f(a, x) \rightarrow a$ satisfies the condition $f(a, x) \succ_{rpo} a$ by the subterm property, as the right-hand side of the rule is a subterm of its left-hand side. Note that, in order to justify the orientation of this rule, it is not necessary to establish a precedence between f and a, due to the fact that a reduction ordering is a simplification ordering.

As regards the second rule, we need to verify $f(h(x), y) \succ_{rpo} g(y, f(x, y))$. Since in the ordering given on Σ we have $f > g$, by rpo definition such a disequality is true iff $\{f(h(x), y)\} \succ\succ_{rpo} \{y, f(x, y)\}$. By the definition of multiset ordering this is true iff the disequalities of the following system are true:

$$f(h(x), y) \succ_{rpo} y$$
$$f(h(x), y) \succ_{rpo} f(x, y)$$

The former is true by the subterm property or the (iv) clause of the generalized rpo. The latter is true iff (by rpo def. with $f = f$) $\{h(x), y\} \succ\!\succ_{rpo} \{x, y\}$ iff (multiset ordering def.) $h(x) \succ_{rpo} x$, which is verified by the subterm property or the (iv) clause of the generalized rpo. Hence, we have shown that $f(h(x), y) \succ_{rpo} g(y, f(x, y))$.

For the third rule of R we need to prove $g(h(a), x) \succ_{rpo} h(x)$. Because in the partial ordering on Σ we have $g > h$, by rpo def. such a disequality is true iff $\{g(h(a), x)\} \succ\!\succ_{rpo} \{x\}$, that is $g(h(a), x) \succ_{rpo} x$, which is true by the subterm property or the (iv) clause of the generalized rpo.

Thus, we have verified that for each rule in R the left-hand side is greater than the right-hand side in the reduction ordering $\succ_{rpo}$ based on $f > g > h$. By Lankford Theorem it follows that R is terminating with respect to such a reduction ordering.

(I.2) *Orient the identities in an equational theory E so that the resulting trs is terminating with respect to a reduction ordering $\succ$.*

(Ex. T3 in [1]) Given a signature $\Sigma = \{a, f, g, k\}$, orient the equations

$$\begin{aligned} g(x, f(g(a, f(y, z)), y)) &= g(x, f(y, z)) \\ g(f(x, y), z) &= f(x, g(y, z)) \\ g(k(x), f(y, x)) &= k(f(x, y)) \end{aligned}$$

so that the resulting rewrite system is terminating with respect to an rpo based on the precedences $k > g > f$. Justify your answer.

Let us check whether $g(x, f(g(a, f(y, z)), y)) \succ_{rpo} g(x, f(y, z))$. As the function symbols at the root of the two terms are equal ($g = g$), by rpo def. the disequality is true iff $\{x, f(g(a, f(y, z)), y)\} \succ\!\succ_{rpo} \{x, f(y, z)\}$ iff (multiset ordering def.) $f(g(a, f(y, z)), y) \succ_{rpo} f(y, z)$ iff (rpo def. with $f = f$) $\{g(a, f(y, z)), y\} \succ\!\succ_{rpo} \{y, z\}$ iff (multiset ordering def.) $g(a, f(y, z)) \succ_{rpo} z$, which is true by the subterm property or the (iv) clause of the generalized rpo. Thus, as we have shown that $g(x, f(g(a, f(y, z)), y)) \succ_{rpo} g(x, f(y, z))$, the first equation is oriented from left to right by obtaining the rule $g(x, f(g(a, f(y, z)), y)) \to g(x, f(y, z))$.

Let us check whether $g(f(x,y),z) \succ_{rpo} f(x,g(y,z))$. As $g > f$, by rpo def. the disequality is true iff $\{g(f(x,y),z)\} \succ\!\succ_{rpo} \{x, g(y,z)\}$ iff (multiset ordering def.) $g(f(x,y),z) \succ_{rpo} x$ (true by the subterm property or the (iv) clause of the generalized rpo) and $g(f(x,y),z) \succ_{rpo} g(y,z)$. The latter is true iff (rpo def. with $g = g$) $\{f(x,y),z\} \succ\!\succ_{rpo} \{y,z\}$ iff (multiset ordering def.) $f(x,y) \succ_{rpo} y$ (true by the subterm property or the (iv) clause of the generalized rpo). Hence, the resulting rule is $g(f(x,y),z) \rightarrow f(x,g(y,z))$.

Finally, let us check whether $g(k(x), f(y,x)) \succ_{rpo} k(f(x,y))$. As $g < k$, by the (iii) clause of rpo def. we should have $\{k(x), f(y,x)\} \succcurlyeq\!\succcurlyeq_{rpo} \{k(f(x,y))\}$, but neither $k(x)$ nor $f(y,x)$ are $\succcurlyeq_{rpo}$ than $k(f(x,y))$. Thus, let us verify whether $k(f(x,y)) \succ_{rpo} g(k(x), f(y,x))$.[1] As $k > g$, by rpo def. we have $\{k(f(x,y))\} \succ\!\succ_{rpo} \{k(x), f(y,x)\}$ iff (multiset ordering def.) the disequalities of the following system are true:

$$\begin{array}{l} k(f(x,y)) \succ_{rpo} k(x) \\ k(f(x,y)) \succ_{rpo} f(y,x) \end{array}$$

The former is true iff (as $k = k$) $\{f(x,y)\} \succ\!\succ_{rpo} \{x\}$ iff $f(x,y) \succ_{rpo} x$ (true by the subterm property or the (iv) clause of the generalized rpo). The latter is true iff (as $k > f$) we have $\{k(f(x,y))\} \succ\!\succ_{rpo} \{y,x\}$ iff $k(f(x,y)) \succ_{rpo} y$ and $k(f(x,y)) \succ_{rpo} x$, which are both true by the subterm property or the (iv) clause of the generalized rpo.

Thus, the trs R terminating with respect to the given rpo is as follows:

$$\begin{array}{rcl} g(x, f(g(a, f(y,z)), y)) & \rightarrow & g(x, f(y,z)) \\ g(f(x,y),z) & \rightarrow & f(x, g(y,z)) \\ k(f(x,y)) & \rightarrow & g(k(x), f(y,x)) \end{array}$$

[1] One cannot simply derive that this is true because the opposite orientation is not true, as in general the ordering on terms is partial, hence the two terms might be incomparable with respect to the given ordering. It follows that it is always necessary to formally check that a term is greater than another one in the given ordering.

(II) *Give an ordering on terms $\succ$ such that a trs R is terminating with respect to $\succ$.*

(Ex. T14 in [1]) Let the following trs R be defined on the signature $\Sigma = \{a, f, g, h\}$:

$$\begin{array}{rcl} g(a, x) & \to & x \\ g(h(x), y) & \to & h(g(x, y)) \\ f(a) & \to & a \\ f(h(a)) & \to & h(a) \\ f(h(h(x))) & \to & g(f(x), f(h(x))) \end{array}$$

Give an ordering on terms such that R is terminating with respect to such an ordering. Justify your answer.

If we take any simplification ordering $\succ$, we can justify the orientation of rules 1, 3 and 4, as $g(a, x) \succ x$, $f(a) \succ a$ and $f(h(a)) \succ h(a)$ by the subterm property. However, a simplification ordering is not sufficient to justify the orientation of rules 2 and 5. Hence, we must consider an rpo $\succ_{rpo}$ and find an ordering $>$ on the signature Σ such that R is terminating with respect to the rpo based over the precedences defined by $>$. Obviously, the three disequalities above for rules 1, 3 and 4 are also valid for $\succ_{rpo}$, as an rpo is a simplification ordering.

Rule 2: we can derive $g(h(x), y) \succ_{rpo} h(g(x, y))$ by applying the rpo def. with the assumption $g > h$ and verifying that $\{g(h(x), y)\} \succ\!\!\succ_{rpo} \{g(x, y)\}$. This is true iff (multiset ordering def.) $g(h(x), y) \succ_{rpo} g(x, y)$ iff (as $g = g$) $\{h(x), y\} \succ\!\!\succ_{rpo} \{x, y\}$ iff $h(x) \succ_{rpo} x$, which is true by the subterm property or the (iv) clause of the generalized rpo.

Rule 5: $f(h(h(x))) \succ_{rpo} g(f(x), f(h(x)))$ if we assume $f > g$ and prove that $\{f(h(h(x)))\} \succ\!\!\succ_{rpo} \{f(x), f(h(x))\}$. This is true iff $f(h(h(x))) \succ_{rpo} f(x)$ and $f(h(h(x))) \succ_{rpo} f(h(x))$. [2] By multiset ordering def. and rpo def. with $f = f$, the first disequality reduces to proving $h(h(x)) \succ_{rpo} x$ (true by the subterm property or the (iv) clause of the generalized rpo), while the second one reduces to proving $h(h(x)) \succ_{rpo} h(x)$ (true by the subterm property).

We conclude that R is terminating with respect to an rpo based over the precedences $f > g > h$.

[2] In the following we might skip some steps in the formal derivations.

(Ex. T1 in [1]) Given the trs R:

$$\begin{array}{rcl} h(z, g(x,y)) & \rightarrow & g(k(x), h(z,y)) \\ g(k(x), k(y)) & \rightarrow & k(g(x,y)) \end{array}$$

give an ordering $>$ on the operators $\{g, h, k\}$ such that the trs R is terminating with respect to the rpo based over the precedences $>$. Motivate your answer.

The text of the exercise already suggests that the term ordering to be determined is an rpo. We need to find the precedences, i.e. the (partial) ordering on the signature $\{g, h, k\}$. In order to get $h(z, g(x,y)) \succ_{rpo} g(k(x), h(z,y))$, we can establish $h > g$ and verify $\{h(z, g(x,y))\} \succ\!\succ_{rpo} \{k(x), h(z,y)\}$. This is true iff (multiset ordering def.)

$$\begin{array}{l} h(z, g(x,y)) \succ_{rpo} k(x) \\ h(z, g(x,y)) \succ_{rpo} h(z,y) \end{array}$$

To derive the first disequality it is enough to set $h > k$, thus resulting in proving $h(z, g(x,y)) \succ_{rpo} x$, that is true by the subterm property or the (iv) clause of the generalized rpo. For the second one we have $h = h$ and $\{z, g(x,y)\} \succ\!\succ_{rpo} \{z, y\}$ iff $g(x,y) \succ_{rpo} y$ (true by the subterm property or the (iv) clause of the generalized rpo).

In order to have $g(k(x), k(y)) \succ_{rpo} k(g(x,y))$ we can set $g > k$ and prove $g(k(x), k(y)) \succ_{rpo} g(x,y)$. As $g = g$, we get $\{k(x), k(y)\} \succ\!\succ_{rpo} \{x, y\}$ iff $k(x) \succ_{rpo} x$ and $k(y) \succ_{rpo} y$, which are both true by the subterm property or the (iv) clause of the generalized rpo.

The trs R is thus terminating with respect to an rpo based on the precedences $h > g$, $h > k$ and $g > k$, that can also be written as $h > g > k$, from which $h > k$ follows by transitivity.

N.B. If a trs R is terminating with respect to a reduction ordering $\succ$, such ordering $\succ$ is not necessarily the only one that makes R terminating.

(III) *Verify that a trs R in not terminating with respect to any rpo.*

(Ex. T4 in [1]) Given the trs R defined on the signature $\Sigma = \{f, g, h, k\}$:

$$
\begin{array}{rcl}
k(h(x)) & \rightarrow & h(k(x)) \\
k(g(x)) & \rightarrow & h(h(g(x))) \\
f(h(x)) & \rightarrow & f(k(x))
\end{array}
$$

show that there is no ordering $>$ on the operators in Σ such that R is terminating with respect to the rpo based on $>$.

Rule 1: $k(h(x)) \succ_{rpo} h(k(x))$ if $k > h$ and $k(h(x)) \succ_{rpo} k(x)$ iff (as $k = k$) $h(x) \succ_{rpo} x$ (true by the subterm property or the (iv) clause of the generalized rpo).
Rule 2: $k(g(x)) \succ_{rpo} h(h(g(x)))$ if $k > h$ and $k(g(x)) \succ_{rpo} h(g(x))$ iff $k > h$ and $k(g(x)) \succ_{rpo} g(x)$ (true by the subterm property).
Rule 3: $f(h(x)) \succ_{rpo} f(k(x))$ iff (as $f = f$) $h(x) \succ_{rpo} k(x)$. Here the only possibility is to have $h > k$, so that we can derive $h(x) \succ_{rpo} x$ by the subterm property or the (iv) clause of the generalized rpo. But $h > k$ is contradicting the assumption $k > h$ made for justifying the orientation of the first two rules. Let us check whether, by assuming $h > k$ instead of $k > h$, we can justify the orientation of the first rule by deriving $k(h(x)) \succ_{rpo} h(k(x))$. By applying the (iii) clause of the rpo def. with $k < h$, we need to prove $\{h(x)\} \succ\!\!\succcurlyeq_{rpo} \{h(k(x))\}$ iff $h(x) \succcurlyeq_{rpo} h(k(x))$ iff (as $h = h$) $x \succcurlyeq_{rpo} k(x)$, which is false. The same reasoning holds for the second rule under the assumption $h > k$. We conclude that it is not possible to provide an ordering $>$ on Σ such that R is terminating with respect to the rpo based on $>$.

References

[1] M. Nesi, Esercizi di Riscrittura,
in `http://www.di.univaq.it/monica/MFI/EserciziR.pdf`.