University of L'Aquila

Master Degree in Computer Science

Course on Formal Methods

Divergence of completion: examples

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Example 1 Let us consider the following equational theory E on the signature $\Sigma = \{f, g\}$ given by only one equation:

$$f(g(f(x))) \quad = \quad g(f(x))$$

Suppose we want to complete E w.r.t. a reduction ordering $\succ$.
Any simplification ordering is sufficient, because by subterm property we have $f(g(f(x))) \succ g(f(x))$. Thus, by applying the inference rule Orient we get the following TRS R:

$$1. \quad f(g(f(x))) \quad \rightarrow \quad g(f(x))$$

Note that the equation of E can only be oriented from left to right, as the opposite orientation results in the non-termination of the rewrite relation.

Using the inference rule Deduce for computing critical pairs and then checking their convergence, we note that rule 1 overlaps on itself. Given a variant $1'$. $f(g(f(y))) \rightarrow g(f(y))$ of rule 1, we obtain:

1. $cp(1, 1')$ on $p = 1.1$ with mgu $\sigma = \{g(f(x))/y\}$

$$\begin{array}{ccc} & f(g(f(g(f(x))))) & \\ & \swarrow \quad \searrow & \\ f(g(g(f(x)))) & & g(f(g(f(x)))) \\ & & \downarrow \\ & & g(g(f(x))) \end{array}$$

The right-hand side of the c.p. can be reduced with rule 1 in position 1 yielding $g(g(f(x)))$, which is a subterm of the left-hand side of the c.p., that is not convergent. Therefore, the following rule is added to the current TRS:

$$2.\quad f(g(g(f(x')))) \rightarrow g(g(f(x')))$$

2. $cp(1,2)$ on $p = 1.1.1$ with mgu $\sigma = \{g(f(x))/x'\}$

$$\begin{array}{ccc} & f(g(g(f(g(f(x)))))) & \\ \swarrow & & \searrow \\ f(g(g(g(f(x))))) & & g(g(f(g(f(x))))) \\ & & \downarrow \\ & & g(g(g(f(x)))) \end{array}$$

Also in this case the right-hand side of the c.p. can be reduced with rule 1 in position 1.1 yielding $g(g(g(f(x))))$, which is a subterm of the left-hand side of the c.p., that is not convergent. Thus, the following rule is added to the current TRS:

$$3.\quad f(g(g(g(f(x''))))) \rightarrow g(g(g(f(x''))))$$

If we go on with the computation of critical pairs and the completion procedure, we observe that rule 1 and the rules derived from critical pairs overlap between each other by generating an infinite chain of rewrite rules. Hence, we get a divergent (i.e., non-terminating) completion, in the sense that the canonical TRS R equivalent to E exists at infinity and is made by an infinite number of rules. In some cases of divergence of completion, the infinite set of rewrite rules can be formalized in a finite way by means of a *divergence pattern*, that is a higher-order rewrite rule or *meta-rule* (according to the approach defined by H. Kirchner) that characterizes an infinite set of rules in a finite manner. In the above example the divergence pattern is the following:

$$\{f(g^n(f(x))) \rightarrow g^n(f(x)) \;\mid\; n \geq 1\}$$

We can verify that for $n=1$ the divergence pattern becomes rule 1, for $n=2$ we have rule 2, etc.

Example 2 Let us consider the exercise C12 in [1].

Given the equational theory E on the signature $\Sigma = \{e, f, g\}$:

$$\begin{array}{rcl} f(x,x) & = & e \\ f(g(x),y) & = & g(f(x,y)) \end{array}$$

show that the completion of E (w.r.t. an rpo $\succ_{rpo}$ based on the precedences $f > g > e$) diverges and derive a divergence pattern.

By applying the inference rule Orient on the first equation we get $f(x,x) \succ_{rpo} e$ by rpo definition, as $f > e$ and $\{f(x,x)\} \succ\!\!\succ_{rpo} \emptyset$.
For the second equation we have $f(g(x),y) \succ_{rpo} g(f(x,y))$, because $f > g$ and $\{f(g(x),y)\} \succ\!\!\succ_{rpo} \{f(x,y)\}$ iff $f(g(x),y) \succ_{rpo} f(x,y)$ iff $f = f$ and $\{g(x),y\} \succ\!\!\succ_{rpo} \{x,y\}$ iff $g(x) \succ_{rpo} x$, true by subterm property. Hence, we have the following TRS R (with the variables of rules suitably renamed):

$$\begin{array}{rcl} f(x_1,x_1) & \to & e \\ f(g(x_2),y_2) & \to & g(f(x_2,y_2)) \end{array}$$

By applying the inference rule Deduce for the computation of critical pairs and then checking their convergence, we get:

1. $cp(1,2)$ on $p = \epsilon$ with mgu $\sigma = \{g(x_2)/x_1,\ g(x_2)/y_2\}$

$$\begin{array}{ccc} & f(g(x_2),g(x_2)) & \\ \swarrow & & \searrow \\ e & & g(f(x_2,g(x_2))) \end{array}$$

The c.p. is not convergent and is in normal form w.r.t. the current TRS. By rpo definition, as $g > e$, we have $g(f(x_2,g(x_2))) \succ_{rpo} e$. Thus, the new rule to be added to the current TRS is:

$$3. \quad g(f(x_3,g(x_3))) \to e$$

2. $cp(2,3)$ on $p = 1$ with mgu $\sigma = \{g(x_2)/x_3,\ g(g(x_2))/y_2\}$

$$
\begin{array}{ccc}
 & g(f(g(x_2), g(g(x_2)))) & \\
 & \swarrow \quad \searrow & \\
g(g(f(x_2, g(g(x_2))))) & & e
\end{array}
$$

This c.p. is also not convergent and is in normal form w.r.t. the current TRS. Since $g > e$, by rpo definition we have $g(g(f(x_2, g(g(x_2))))) \succ_{rpo} e$. Thus, the new rule to be added to the current TRS is:

$$4. \quad g(g(f(x_4, g(g(x_4))))) \to e$$

3. $cp(3, 2)$ on $p = 1$ with mgu $\sigma = \{f(x_3, g(x_3))/x_2\}$

$$
\begin{array}{ccc}
 & f(g(f(x_3, g(x_3))), y_2) & \\
 & \swarrow \quad \searrow & \\
f(e, y_2) & & g(f(f(x_3, g(x_3)), y_2))
\end{array}
$$

The c.p. is not convergent and is in normal form w.r.t. the current TRS. In order to orient the c.p. we show that $g(f(f(x_3, g(x_3)), y_2)) \succ_{rpo} f(e, y_2)$. As $g < f$, by rpo definition we have $\{f(f(x_3, g(x_3)), y_2)\} \succ\!\succeq_{rpo} \{f(e, y_2)\}$ iff $f(f(x_3, g(x_3)), y_2) \succeq_{rpo} f(e, y_2)$ iff $f = f$ and $\{f(x_3, g(x_3)), y_2\} \succ\!\succeq_{rpo} \{e, y_2\}$ iff $f(x_3, g(x_3)) \succeq_{rpo} e$ iff $f > e$ and $\{f(x_3, g(x_3))\} \succ\!\succeq_{rpo} \emptyset$, true by the definition of $\succ\!\succeq_{rpo}$. Thus, the new rule to be added to the current TRS is:

$$5. \quad g(f(f(x_5, g(x_5)), y_5)) \to f(e, y_5)$$

4. $cp(2, 4)$ on $p = 1.1$ with mgu $\sigma = \{g(x_2)/x_4,\ g(g(g(x_2)))/y_2\}$

$$
\begin{array}{ccc}
 & g(g(f(g(x_2), g(g(g(x_2)))))) & \\
 & \swarrow \quad \searrow & \\
g(g(g(f(x_2, g(g(g(x_2))))))) & & e
\end{array}
$$

Similarly to the previous critical pairs, we have that the c.p. is not convergent and is in normal form w.r.t. the current TRS. As $g > e$, by rpo definition we have $g(g(g(f(x_2, g(g(g(x_2))))))) \succ_{rpo} e$. Thus, the new rule to be added to the current TRS is:

$$6. \quad g(g(g(f(x_6, g(g(g(x_6))))))) \to e$$

If we continue with the computation of critical pairs and the completion procedure, we have that the rules of the current TRS overlap between each other by generating an infinite chain of rewrite rules. The completion diverges and the following is a divergence pattern:

$$\{g^n(f(x, g^n(x))) \to e \quad | \quad n \geq 1\}$$

We can verify that for $n = 1$ the divergence pattern becomes rule 3, for $n = 2$ we have rule 4, for $n = 3$ we get rule 6, etc.

References

[1] M. Nesi, Esercizi di Riscrittura,
in `http://www.di.univaq.it/monica/MFI/EserciziR.pdf`.