

**University of L'Aquila**  
**Master Degree in Computer Science**  
**Course on Formal Methods**  
**An exercise of completion of equational theories**  
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Let us consider an equational presentation of a fragment of the group theory taken from [DJ90], that is particularly interesting from the point of view of completion because, when applying the completion procedure on such a theory, non-basic inference rules can also be applied, such as Collapse, and equations can be deleted using the inference rule Delete. The given equational theory  $E$  (where a prefixed notation is used) is the following:

$$\begin{aligned}\bullet(x, 1) &= x \\ \bullet(1, x) &= x \\ \bullet(I(x), \bullet(x, y)) &= y\end{aligned}$$

On a signature  $\Sigma = \{1, I, \bullet\}$ , the first two equations in  $E$  state that the constant 1 is a neutral element, both on the right and on the left, for the operator  $\bullet$ . The third equation asserts that, if the inverse (or opposite)  $I(x)$  of  $x$  is combined with  $x$  itself and any  $y$  using  $\bullet$ , then the resulting term is equivalent to  $y$ .

Suppose we want to complete  $E$  with respect to an rpo based on precedence  $\bullet > 1$ . By applying the rule Orient three times to orient the three equations in  $E$  and suitably renaming the variables, the following TRS  $R$  is obtained:

$$\begin{aligned}1. \quad & \bullet(x_1, 1) \quad \rightarrow \quad x_1 \\ 2. \quad & \bullet(1, x_2) \quad \rightarrow \quad x_2 \\ 3. \quad & \bullet(I(x_3), \bullet(x_3, y_3)) \quad \rightarrow \quad y_3\end{aligned}$$

Note that, in order to orient the three equations in  $E$ , any simplification ordering would be enough, as for each rule we have that the left-hand side is greater than the right-hand side by subterm property. However, a simplification ordering would not be sufficient to orient the non-convergent critical pairs that will be generated during the completion.

By applying the rule Deduce for the computation of critical pairs and then verifying their convergence, we have:

1.  $cp(1, 2)$  on  $p = \epsilon$  with mgu  $\sigma = \{1/x_1, 1/x_2\}$

$$\begin{array}{c} \bullet(1, 1) \\ \swarrow \quad \searrow \\ 1 \equiv 1 \end{array}$$

The c.p. is trivially convergent, as the two terms of the c.p. are the same, thus it is deleted using Delete.

2.  $cp(1, 3)$  on  $p = 2$  with mgu  $\sigma = \{x_1/x_3, 1/y_3\}$

$$\begin{array}{c} \bullet(I(x_1), \bullet(x_1, 1)) \\ \swarrow \quad \searrow \\ \bullet(I(x_1), x_1) \quad 1 \end{array}$$

The c.p. is not convergent, thus it is an equation to be oriented into a rewrite rule through Orient. As  $\bullet > 1$ , by rpo definition we have  $\bullet(I(x_1), x_1) \succ_{rpo} 1$ . Hence, the new rule to be added to the current TRS  $R$  (with variables suitably renamed) is:

$$4. \quad \bullet(I(x_4), x_4) \rightarrow 1$$

3.  $cp(2, 3)$  on  $p = 2$  with mgu  $\sigma = \{1/x_3, x_2/y_3\}$

$$\begin{array}{c} \bullet(I(1), \bullet(1, x_2)) \\ \swarrow \quad \searrow \\ \bullet(I(1), x_2) \quad x_2 \end{array}$$

The c.p. is not convergent and its two terms are normalized w.r.t. the current TRS. By subterm property  $\bullet(I(1), x_2) \succ_{rpo} x_2$ , thus applying the inference rule Orient yields the new rule:

$$5. \quad \bullet(I(1), x_5) \rightarrow x_5$$

4.  $cp(3, 3)$  on  $p = 2$  with mgu  $\sigma = \{I(x_3)/x, \bullet(x_3, y_3)/y\}$

$$\begin{array}{c} \bullet(I(I(x_3)), \bullet(I(x_3), \bullet(x_3, y_3))) \\ \swarrow \quad \searrow \\ \bullet(I(I(x_3)), y_3) \quad \bullet(x_3, y_3) \end{array}$$

The c.p. is not convergent and its two terms are normalized w.r.t. the current TRS. By applying the rpo definition we have:

$\bullet(I(I(x_3)), y_3) \succ_{rpo} \bullet(x_3, y_3)$  iff  $\{I(I(x_3)), y_3\} \succ_{rpo} \{x_3, y_3\}$  iff  $I(I(x_3)) \succ_{rpo} x_3$ , true by subterm property.

Hence, the new rule to be added to the current TRS  $R$  is:

$$6. \quad \bullet(I(I(x_6)), y_6) \rightarrow \bullet(x_6, y_6)$$

5.  $cp(1, 4)$  on  $p = \epsilon$  with mgu  $\sigma = \{I(1)/x_1, 1/x_4\}$

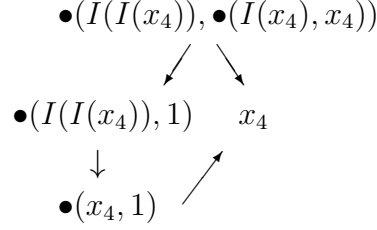
$$\begin{array}{c} \bullet(I(1), 1) \\ \swarrow \quad \searrow \\ I(1) \quad 1 \end{array}$$

The c.p. is not convergent and, since  $I(1) \succ_{rpo} 1$  by subterm property, we get the new rule:

$$7. \quad I(1) \rightarrow 1$$

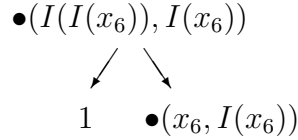
With the introduction of rule 7 we can apply the inference rule Collapse on rule 5, as the left-hand side of rule 5 can be reduced with rule 7 to the term  $\bullet(1, x_5)$ . The resulting equation  $\bullet(1, x_5) = x_5$  can be reduced to  $x_5 = x_5$  (Simplify on the left with rule 2) and then removed (Delete). Therefore, rule 5 is not in the current TRS  $R$  anymore.

6.  $cp(4, 3)$  on  $p = 2$  with mgu  $\sigma = \{I(x_4)/x_3, x_4/y_3\}$



The left-hand side of the c.p. can be reduced using rule 6, thus obtaining  $\bullet(x_4, 1)$  that rewrites to the right-hand side of the c.p. using rule 1. Hence, the c.p. is convergent and no new rule is added in  $R$ . In terms of the completion inference rules this means to apply Simplify on the left and then Delete on the equation derived from critical pair.

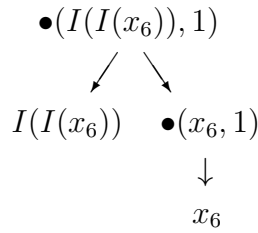
7.  $cp(4, 6)$  on  $p = \epsilon$  with mgu  $\sigma = \{I(x_6)/x_4, I(x_6)/y_6\}$



The c.p. is not convergent and, since  $\bullet > 1$ , by the rpo definition we have  $\bullet(x_6, I(x_6)) \succ_{rpo} 1$ , thus we add the new rule:

$$8. \quad \bullet(x_8, I(x_8)) \rightarrow 1$$

8.  $cp(1, 6)$  on  $p = \epsilon$  with mgu  $\sigma = \{I(I(x_6))/x_1, 1/y_6\}$



The right-hand side of the c.p. reduces to  $x_6$  using rule 1. The c.p. is not convergent and by subterm property  $I(I(x_6)) \succ_{rpo} x_6$ , thus getting the new rule:

$$9. \quad I(I(x_6)) \rightarrow x_6$$

With the introduction of rule 9 we can apply the inference rule Collapse on rule 6, as the left-hand side of rule 6 can be reduced with rule 9 to the term  $\bullet(x_6, y_6)$ . The resulting equation  $\bullet(x_6, y_6) = \bullet(x_6, y_6)$  is removed using Delete. Thus, rule 6 is not in the current TRS  $R$  anymore.

9.  $cp(7, 3)$  on  $p = 1$  with mgu  $\sigma = \{1/x_3\}$

$$\begin{array}{c} \bullet(I(1), \bullet(1, y_3)) \\ \swarrow \quad \searrow \\ \bullet(1, \bullet(1, y_3)) \xrightarrow{+} y_3 \end{array}$$

10.  $cp(7, 4)$  on  $p = 1$  with mgu  $\sigma = \{1/x_4\}$

$$\begin{array}{c} \bullet(I(1), 1) \\ \swarrow \quad \searrow \\ \bullet(1, 1) \rightarrow 1 \end{array}$$

11.  $cp(7, 8)$  on  $p = 2$  with mgu  $\sigma = \{1/x_8\}$

$$\begin{array}{c} \bullet(1, I(1)) \\ \swarrow \quad \searrow \\ \bullet(1, 1) \rightarrow 1 \end{array}$$

12.  $cp(7, 9)$  on  $p = 1$  with mgu  $\sigma = \{1/x_9\}$

$$\begin{array}{c} I(I(1)) \\ \swarrow \quad \searrow \\ I(1) \rightarrow 1 \end{array}$$

13.  $cp(2, 8)$  on  $p = \epsilon$  with mgu  $\sigma = \{1/x_8, I(1)/x_2\}$

$$\begin{array}{c} \bullet(1, I(1)) \\ \swarrow \quad \searrow \\ I(1) \rightarrow 1 \end{array}$$

14.  $cp(8, 3)$  on  $p = 2$  with mgu  $\sigma = \{x_8/x_3, I(x_8)/y_3\}$

$$\begin{array}{c} \bullet(I(x_8), \bullet(x_8, I(x_8))) \\ \swarrow \quad \searrow \\ \bullet(I(x_8), 1) \rightarrow I(x_8) \end{array}$$

15.  $cp(9, 9)$  on  $p = 1$  with mgu  $\sigma = \{I(x)/x_9\}$

$$\begin{array}{c} I(I(I(x))) \\ \swarrow \quad \searrow \\ I(x) \equiv I(x) \end{array}$$

16.  $cp(9, 3)$  on  $p = \epsilon$  with mgu  $\sigma = \{I(x_9)/x_3\}$

$$\begin{array}{c} \bullet(I(I(x_9)), \bullet(I(x_9), y_3)) \\ \swarrow \quad \searrow \\ \bullet(x_9, \bullet(I(x_9), y_3)) \quad y_3 \end{array}$$

The c.p. is not convergent. As  $\bullet(x_9, \bullet(I(x_9), y_3)) \succ_{rpo} y_3$  by subterm property, the new rule is introduced:

$$10. \quad \bullet(x_{10}, \bullet(I(x_{10}), y_{10})) \rightarrow y_{10}$$

17.  $cp(9, 4)$  on  $p = 1$  with mgu  $\sigma = \{I(x_9)/x_4\}$

$$\begin{array}{c} \bullet(I(I(x_9)), I(x_9)) \\ \swarrow \quad \searrow \\ \bullet(x_9, I(x_9)) \rightarrow 1 \end{array}$$

18.  $cp(9, 8)$  on  $p = 2$  with mgu  $\sigma = \{I(x_9)/x_8\}$

$$\begin{array}{c} \bullet(I(x_9), I(I(x_9))) \\ \swarrow \quad \searrow \\ \bullet(I(x_9), x_9) \rightarrow 1 \end{array}$$

19.  $cp(1, 10)$  on  $p = 2$  with mgu  $\sigma = \{I(x_{10})/x_1, 1/y_{10}\}$

$$\begin{array}{c} \bullet(x_{10}, \bullet(I(x_{10}), 1)) \\ \swarrow \quad \searrow \\ \bullet(x_{10}, I(x_{10})) \rightarrow 1 \end{array}$$

20.  $cp(2, 10)$  on  $p = \epsilon$  with mgu  $\sigma = \{1/x_{10}, \bullet(I(1), y_{10})/x_2\}$

$$\begin{array}{c} \bullet(1, \bullet(I(1), y_{10})) \\ \swarrow \quad \searrow \\ \bullet(I(1), y_{10}) \quad y_{10} \\ \downarrow \quad \nearrow \\ \bullet(1, y_{10}) \end{array}$$

21.  $cp(3, 10)$  on  $p = 2$  with mgu  $\sigma = \{x_3/x_{10}, \bullet(x_3, y_3)/y_{10}\}$

$$\begin{array}{c} \bullet(x_3, \bullet(I(x_3), \bullet(x_3, y_3))) \\ \swarrow \quad \searrow \\ \bullet(x_3, y_3) \equiv \bullet(x_3, y_3) \end{array}$$

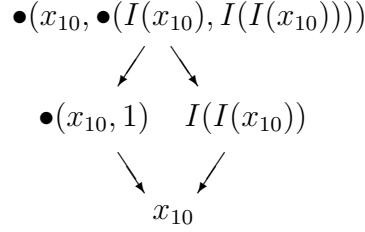
22.  $cp(4, 10)$  on  $p = 2$  with mgu  $\sigma = \{x_4/x_{10}, x_4/y_{10}\}$

$$\begin{array}{c} \bullet(x_4, \bullet(I(x_4), x_4)) \\ \swarrow \quad \searrow \\ \bullet(x_4, 1) \rightarrow x_4 \end{array}$$

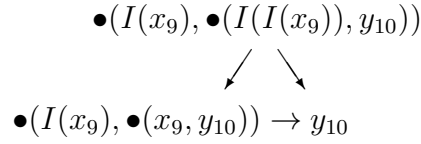
23.  $cp(7, 10)$  on  $p = 2.1$  with mgu  $\sigma = \{1/x_{10}\}$

$$\begin{array}{c} \bullet(1, \bullet(I(1), y_{10})) \\ \swarrow \quad \searrow \\ \bullet(1, \bullet(1, y_{10})) \xrightarrow{+} y_{10} \end{array}$$

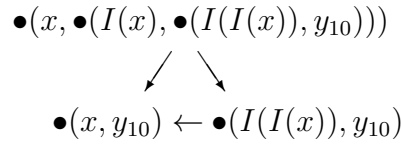
24.  $cp(8, 10)$  on  $p = 2$  with mgu  $\sigma = \{I(x_{10})/x_8, I(I(x_{10}))/y_{10}\}$



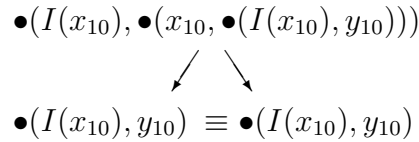
25.  $cp(9, 10)$  on  $p = 2.1$  with mgu  $\sigma = \{I(x_9)/x_{10}\}$



26.  $cp(10, 10)$  on  $p = 2$  with mgu  $\sigma = \{I(x)/x_{10}, \bullet(I(I(x)), y_{10})/y\}$



27.  $cp(10, 3)$  on  $p = 2$  with mgu  $\sigma = \{x_{10}/x_3, \bullet(I(x_{10}), y_{10})/y_3\}$



There are no other critical pairs. The set of equations is empty and the resulting TRS  $R$  is terminating and locally confluent, hence is also confluent and canonical. The TRS  $R$  is as follows:



$$\begin{aligned}
& \bullet(x, 1) \rightarrow x \\
& \bullet(1, x) \rightarrow x \\
& \bullet(I(x), \bullet(x, y)) \rightarrow y \\
& \bullet(I(x), x) \rightarrow 1 \\
& I(1) \rightarrow 1 \\
& \bullet(x, I(x)) \rightarrow 1 \\
& I(I(x)) \rightarrow x \\
& \bullet(x, \bullet(I(x), y)) \rightarrow y
\end{aligned}$$

Note that during the completion the rules of  $R$  can be generated in a different order depending on the strategy used when choosing the rules for deducing critical pairs and for reducing them. For example, given the c.p. (6), if its left-hand side is reduced by applying rule 1 (instead of rule 6) we obtain rule 9 straightaway.

## References

[DJ90] N. Dershowitz and J.-P. Jouannaud, ‘Rewrite Systems’, in *Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics*, J. van Leeuwen editor, North-Holland, 1990, pp. 243-320.