University of L'Aquila

Master Degree in Computer Science

Course on Formal Methods

An exercise of completion of equational theories

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Let us consider an equational presentation of a fragment of the group theory taken from [DJ90], that is particularly interesting from the point of view of completion because, when applying the completion procedure on such a theory, non-basic inference rules can also be applied, such as Collapse, and equations can be deleted using the inference rule Delete. The given equational theory E (where a prefixed notation is used) is the following:

$$\begin{array}{rcl} \bullet(x,1) & = & x \\ \bullet(1,x) & = & x \\ \bullet(I(x),\bullet(x,y)) & = & y \end{array}$$

On a signature $\Sigma = \{1, I, \bullet\}$, the first two equations in E state that the constant 1 is a neutral element, both on the right and on the left, for the operator $\bullet$. The third equation asserts that, if the inverse (or opposite) $I(x)$ of x is combined with x itself and any y using $\bullet$, then the resulting term is equivalent to y.

Suppose we want to complete E with respect to an rpo based on precedence $\bullet > 1$. By applying the rule Orient three times to orient the three equations in E and suitably renaming the variables, the following TRS R is obtained:

$$\begin{array}{lrcl} 1. & \bullet(x_1,1) & \to & x_1 \\ 2. & \bullet(1,x_2) & \to & x_2 \\ 3. & \bullet(I(x_3),\bullet(x_3,y_3)) & \to & y_3 \end{array}$$

Note that, in order to orient the three equations in E, any simplification ordering would be enough, as for each rule we have that the left-hand side is greater than the right-hand side by subterm property. However, a simplification ordering would not be sufficient to orient the non-convergent critical pairs that will be generated during the completion.

By applying the rule Deduce for the computation of critical pairs and then verifying their convergence, we have:

1. $cp(1,2)$ on $p = \epsilon$ with mgu $\sigma = \{1/x_1,\ 1/x_2\}$

$$\begin{array}{ccc} & \bullet(1,1) & \\ \swarrow & & \searrow \\ 1 & \equiv & 1 \end{array}$$

The c.p. is trivially convergent, as the two terms of the c.p. are the same, thus it is deleted using Delete.

2. $cp(1,3)$ on $p = 2$ with mgu $\sigma = \{x_1/x_3,\ 1/y_3\}$

$$\begin{array}{ccc} & \bullet(I(x_1), \bullet(x_1, 1)) & \\ \swarrow & & \searrow \\ \bullet(I(x_1), x_1) & & 1 \end{array}$$

The c.p. is not convergent, thus it is an equation to be oriented into a rewrite rule through Orient. As $\bullet > 1$, by rpo definition we have $\bullet(I(x_1), x_1) \succ_{rpo} 1$. Hence, the new rule to be added to the current TRS R (with variables suitably renamed) is:

$$4. \quad \bullet\,(I(x_4), x_4) \to 1$$

3. $cp(2,3)$ on $p = 2$ with mgu $\sigma = \{1/x_3,\ x_2/y_3\}$

$$\begin{array}{ccc} & \bullet(I(1), \bullet(1, x_2)) & \\ \swarrow & & \searrow \\ \bullet(I(1), x_2) & & x_2 \end{array}$$

The c.p. is not convergent and its two terms are normalized w.r.t. the current TRS. By subterm property $\bullet(I(1), x_2) \succ_{rpo} x_2$, thus applying the inference rule Orient yields the new rule:

$$5. \quad \bullet\,(I(1), x_5) \to x_5$$

4. $cp(3,3)$ on $p = 2$ with mgu $\sigma = \{I(x_3)/x,\ \bullet(x_3, y_3)/y\}$

$$\bullet(I(I(x_3)), \bullet(I(x_3), \bullet(x_3, y_3)))$$

$$\swarrow \quad \searrow$$

$$\bullet(I(I(x_3)), y_3) \qquad \bullet(x_3, y_3)$$

The c.p. is not convergent and its two terms are normalized w.r.t. the current TRS. By applying the rpo definition we have:
$\bullet(I(I(x_3)), y_3) \succ_{rpo} \bullet(x_3, y_3)$ iff $\{I(I(x_3)), y_3\} \succ\!\!\succ_{rpo} \{x_3, y_3\}$ iff
$I(I(x_3)) \succ_{rpo} x_3$, true by subterm property.
Hence, the new rule to be added to the current TRS R is:

$$6. \quad \bullet\,(I(I(x_6)), y_6) \to \bullet(x_6, y_6)$$

5. $cp(1,4)$ on $p = \epsilon$ with mgu $\sigma = \{I(1)/x_1,\ 1/x_4\}$

$$\bullet(I(1), 1)$$

$$\swarrow \quad \searrow$$

$$I(1) \qquad 1$$

The c.p. is not convergent and, since $I(1) \succ_{rpo} 1$ by subterm property, we get the new rule:

$$7. \quad I(1) \to 1$$

With the introduction of rule 7 we can apply the inference rule Collapse on rule 5, as the left-hand side of rule 5 can be reduced with rule 7 to the term $\bullet(1, x_5)$. The resulting equation $\bullet(1, x_5) = x_5$ can be reduced to $x_5 = x_5$ (Simplify on the left with rule 2) and then removed (Delete). Therefore, rule 5 is not in the current TRS R anymore.

6. $cp(4,3)$ on $p = 2$ with mgu $\sigma = \{I(x_4)/x_3,\ x_4/y_3\}$

$$\begin{array}{ccc} & \bullet(I(I(x_4)), \bullet(I(x_4), x_4)) & \\ & \swarrow \quad \searrow & \\ \bullet(I(I(x_4)), 1) & & x_4 \\ \downarrow & & \\ \bullet(x_4, 1) & \nearrow & \end{array}$$

The left-hand side of the c.p. can be reduced using rule 6, thus obtaining $\bullet(x_4, 1)$ that rewrites to the right-hand side of the c.p. using rule 1. Hence, the c.p. is convergent and no new rule is added in R. In terms of the completion inference rules this means to apply Simplify on the left and then Delete on the equation derived from critical pair.

7. $cp(4,6)$ on $p = \epsilon$ with mgu $\sigma = \{I(x_6)/x_4,\ I(x_6)/y_6\}$

$$\begin{array}{ccc} & \bullet(I(I(x_6)), I(x_6)) & \\ & \swarrow \quad \searrow & \\ 1 & & \bullet(x_6, I(x_6)) \end{array}$$

The c.p. is not convergent and, since $\bullet > 1$, by the rpo definition we have $\bullet(x_6, I(x_6)) \succ_{rpo} 1$, thus we add the new rule:

$$8. \quad \bullet\,(x_8, I(x_8)) \to 1$$

8. $cp(1,6)$ on $p = \epsilon$ with mgu $\sigma = \{I(I(x_6))/x_1,\ 1/y_6\}$

$$\begin{array}{ccc} & \bullet(I(I(x_6)), 1) & \\ & \swarrow \quad \searrow & \\ I(I(x_6)) & & \bullet(x_6, 1) \\ & & \downarrow \\ & & x_6 \end{array}$$

The right-hand side of the c.p. reduces to x_6 using rule 1. The c.p. is not convergent and by subterm property $I(I(x_6)) \succ_{rpo} x_6$, thus getting the new rule:

$$9. \quad I(I(x_6)) \to x_6$$

With the introduction of rule 9 we can apply the inference rule Collapse on rule 6, as the left-hand side of rule 6 can be reduced with rule 9 to the term $\bullet(x_6, y_6)$. The resulting equation $\bullet(x_6, y_6) = \bullet(x_6, y_6)$ is removed using Delete. Thus, rule 6 is not in the current TRS R anymore.

9. $cp(7, 3)$ on $p = 1$ with mgu $\sigma = \{1/x_3\}$

$$\begin{array}{ccc} & \bullet(I(1), \bullet(1, y_3)) & \\ \swarrow & & \searrow \\ \bullet(1, \bullet(1, y_3)) & \xrightarrow{+} & y_3 \end{array}$$

10. $cp(7, 4)$ on $p = 1$ with mgu $\sigma = \{1/x_4\}$

$$\begin{array}{ccc} & \bullet(I(1), 1) & \\ \swarrow & & \searrow \\ \bullet(1, 1) & \rightarrow & 1 \end{array}$$

11. $cp(7, 8)$ on $p = 2$ with mgu $\sigma = \{1/x_8\}$

$$\begin{array}{ccc} & \bullet(1, I(1)) & \\ \swarrow & & \searrow \\ \bullet(1, 1) & \rightarrow & 1 \end{array}$$

12. $cp(7, 9)$ on $p = 1$ with mgu $\sigma = \{1/x_9\}$

$$\begin{array}{ccc} & I(I(1)) & \\ \swarrow & & \searrow \\ I(1) & \rightarrow & 1 \end{array}$$

13. $cp(2, 8)$ on $p = \epsilon$ with mgu $\sigma = \{1/x_8,\ I(1)/x_2\}$

$$\begin{array}{ccc} & \bullet(1, I(1)) & \\ \swarrow & & \searrow \\ I(1) & \rightarrow & 1 \end{array}$$

14. $cp(8,3)$ on $p = 2$ with mgu $\sigma = \{x_8/x_3,\ I(x_8)/y_3\}$

$$\begin{array}{c} \bullet(I(x_8), \bullet(x_8, I(x_8))) \\ \swarrow \quad \searrow \\ \bullet(I(x_8), 1) \rightarrow I(x_8) \end{array}$$

15. $cp(9,9)$ on $p = 1$ with mgu $\sigma = \{I(x)/x_9\}$

$$\begin{array}{c} I(I(I(x))) \\ \swarrow \quad \searrow \\ I(x) \ \equiv \ I(x) \end{array}$$

16. $cp(9,3)$ on $p = \epsilon$ with mgu $\sigma = \{I(x_9)/x_3\}$

$$\begin{array}{c} \bullet(I(I(x_9)), \bullet(I(x_9), y_3)) \\ \swarrow \quad \searrow \\ \bullet(x_9, \bullet(I(x_9), y_3)) \qquad y_3 \end{array}$$

The c.p. is not convergent. As $\bullet(x_9, \bullet(I(x_9), y_3)) \succ_{rpo} y_3$ by subterm property, the new rule is introduced:

$$10. \quad \bullet\,(x_{10}, \bullet(I(x_{10}), y_{10})) \rightarrow y_{10}$$

17. $cp(9,4)$ on $p = 1$ with mgu $\sigma = \{I(x_9)/x_4\}$

$$\begin{array}{c} \bullet(I(I(x_9)), I(x_9)) \\ \swarrow \quad \searrow \\ \bullet(x_9, I(x_9)) \rightarrow \ 1 \end{array}$$

18. $cp(9,8)$ on $p = 2$ with mgu $\sigma = \{I(x_9)/x_8\}$

$$\begin{array}{c} \bullet(I(x_9), I(I(x_9))) \\ \swarrow \quad \searrow \\ \bullet(I(x_9), x_9) \rightarrow \ 1 \end{array}$$

19. $cp(1,10)$ on $p = 2$ with mgu $\sigma = \{I(x_{10})/x_1,\ 1/y_{10}\}$

$$\begin{array}{c} \bullet(x_{10}, \bullet(I(x_{10}), 1)) \\ \swarrow \quad \searrow \\ \bullet(x_{10}, I(x_{10})) \rightarrow 1 \end{array}$$

20. $cp(2,10)$ on $p = \epsilon$ with mgu $\sigma = \{1/x_{10}, \bullet(I(1), y_{10})/x_2\}$

$$\begin{array}{cc} \multicolumn{2}{c}{\bullet(1, \bullet(I(1), y_{10}))} \\ \swarrow & \searrow \\ \bullet(I(1), y_{10}) & y_{10} \\ \downarrow & \nearrow \\ \bullet(1, y_{10}) & \end{array}$$

21. $cp(3,10)$ on $p = 2$ with mgu $\sigma = \{x_3/x_{10},\ \bullet(x_3, y_3)/y_{10}\}$

$$\begin{array}{c} \bullet(x_3, \bullet(I(x_3), \bullet(x_3, y_3))) \\ \swarrow \quad \searrow \\ \bullet(x_3, y_3) \ \equiv \ \bullet(x_3, y_3) \end{array}$$

22. $cp(4,10)$ on $p = 2$ with mgu $\sigma = \{x_4/x_{10},\ x_4/y_{10}\}$

$$\begin{array}{c} \bullet(x_4, \bullet(I(x_4), x_4)) \\ \swarrow \quad \searrow \\ \bullet(x_4, 1) \rightarrow x_4 \end{array}$$

23. $cp(7,10)$ on $p = 2.1$ with mgu $\sigma = \{1/x_{10}\}$

$$\begin{array}{c} \bullet(1, \bullet(I(1), y_{10})) \\ \swarrow \quad \searrow \\ \bullet(1, \bullet(1, y_{10})) \xrightarrow{+} y_{10} \end{array}$$

24. $cp(8,10)$ on $p = 2$ with mgu $\sigma = \{I(x_{10})/x_8,\ I(I(x_{10}))/y_{10}\}$

$$\begin{array}{ccc} & \bullet(x_{10}, \bullet(I(x_{10}), I(I(x_{10})))) & \\ \swarrow & & \searrow \\ \bullet(x_{10}, 1) & & I(I(x_{10})) \\ \searrow & & \swarrow \\ & x_{10} & \end{array}$$

25. $cp(9,10)$ on $p = 2.1$ with mgu $\sigma = \{I(x_9)/x_{10}\}$

$$\begin{array}{ccc} & \bullet(I(x_9), \bullet(I(I(x_9)), y_{10})) & \\ \swarrow & & \searrow \\ \bullet(I(x_9), \bullet(x_9, y_{10})) & \rightarrow & y_{10} \end{array}$$

26. $cp(10,10)$ on $p = 2$ with mgu $\sigma = \{I(x)/x_{10},\ \bullet(I(I(x)), y_{10})/y\}$

$$\begin{array}{ccc} & \bullet(x, \bullet(I(x), \bullet(I(I(x)), y_{10}))) & \\ \swarrow & & \searrow \\ \bullet(x, y_{10}) & \leftarrow & \bullet(I(I(x)), y_{10}) \end{array}$$

27. $cp(10,3)$ on $p = 2$ with mgu $\sigma = \{x_{10}/x_3,\ \bullet(I(x_{10}), y_{10})/y_3\}$

$$\begin{array}{ccc} & \bullet(I(x_{10}), \bullet(x_{10}, \bullet(I(x_{10}), y_{10}))) & \\ \swarrow & & \searrow \\ \bullet(I(x_{10}), y_{10}) & \equiv & \bullet(I(x_{10}), y_{10}) \end{array}$$

There are no other critical pairs. The set of equations is empty and the resulting TRS R is terminating and locally confluent, hence is also confluent and canonical. The TRS R is as follows:

$$
\begin{array}{rcl}
\bullet(x,1) & \to & x \\
\bullet(1,x) & \to & x \\
\bullet(I(x),\bullet(x,y)) & \to & y \\
\bullet(I(x),x) & \to & 1 \\
I(1) & \to & 1 \\
\bullet(x,I(x)) & \to & 1 \\
I(I(x)) & \to & x \\
\bullet(x,\bullet(I(x),y)) & \to & y
\end{array}
$$

Note that during the completion the rules of R can be generated in a different order depending on the strategy used when choosing the rules for deducing critical pairs and for reducing them. For example, given the c.p. (6), if its left-hand side is reduced by applying rule 1 (instead of rule 6) we obtain rule 9 straightaway.

References

[DJ90] N. Dershowitz and J.-P. Jouannaud, 'Rewrite Systems', in *Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics*, J. van Leeuwen editor, North-Holland, 1990, pp. 243-320.