University of L'Aquila

Master Degree in Computer Science

Course on Formal Methods

Critical Pairs: computation and convergence

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Consider Exercise C3 in [1].
Given the rewrite system R on a signature $\Sigma = \{a, f, g\}$:

$$\begin{array}{rcl} f(f(x,y),z) & \rightarrow & f(x,f(y,z)) \\ f(g(x,y),z) & \rightarrow & g(f(x,z),f(y,z)) \\ g(y,y) & \rightarrow & y \\ f(a,x) & \rightarrow & x \end{array}$$

verify that R is locally confluent.

Note that the rules in R represent the following properties: the first rule (r1) expresses left-to-right associativity for f, the second rule (r2) is the distributivity of f with respect to g, the third rule (r3) represents idempotency of g and the last rule (r4) states the existence of a left neutral element (or identity) for f.

Based on the Huet Lemma, verifying that R is locally confluent means verifying that all critical pairs of R are convergent. When computing critical pairs, the intersection of the sets of variables in the rewrite rules must be empty (see the definition of critical pairs). To satisfy such hypothesis it is enough to suitably rename the variables in the rules of R. For example, R becomes the following system:

$$\begin{array}{rcl} f(f(x_1,y_1),z_1) & \rightarrow & f(x_1,f(y_1,z_1)) \\ f(g(x_2,y_2),z_2) & \rightarrow & g(f(x_2,z_2),f(y_2,z_2)) \\ g(y_3,y_3) & \rightarrow & y_3 \\ f(a,x_4) & \rightarrow & x_4 \end{array}$$

When computing the critical pairs of R we must also check if a rule can overlap on itself in an internal position $p \neq \epsilon$. To have that, it is necessary that the topmost symbol of the left-hand side of a rule also occurs inside the same left-hand side. This condition is necessary, but in general is not sufficient.[1] For example, rule r1 overlaps on itself in position $p=1$. By considering two variants of such rule, a critical pair is computed as follows:

1. $cc(r1, r1) \qquad p=1 \qquad \sigma = \{f(x_1, y_1)/x,\ z_1/y\}$

$$\begin{array}{ccc} & f(f(f(x_1,y_1),z_1),z) & \\ \swarrow & & \searrow \\ f(f(x_1,f(y_1,z_1)),z) & & f(f(x_1,y_1),f(z_1,z)) \end{array}$$

The c.p. is given by the two terms obtained by rewriting the most general term (at the top of the local confluence peak) by rule r1 in positions $p=1$ and $p=\epsilon$. To verify the convergence of the c.p., we check whether the two terms of the c.p. can be rewritten in R to a common term. Possible reductions are the following:

$$f(f(x_1,f(y_1,z_1)),z) \to_{r1,p=\epsilon} f(x_1,f(f(y_1,z_1),z)) \to_{r1,p=2} f(x_1,f(y_1,f(z_1,z)))$$
$$f(f(x_1,y_1),f(z_1,z)) \to_{r1,p=\epsilon} f(x_1,f(y_1,f(z_1,z)))$$

Thus the c.p. is convergent. Rule r1 does not overlap on other rules, that is its left-hand side does not unify with any subterm of the other left-hand sides in R.

Let us consider rule r2. This rule is not overlapping on itself, but overlaps on r1 in position $p=1$:

2. $cc(r2, r1) \qquad p=1 \qquad \sigma = \{g(x_2, y_2)/x_1,\ z_2/y_1\}$

$$\begin{array}{ccc} & f(f(g(x_2,y_2),z_2),z_1) & \\ \swarrow & & \searrow \\ f(g(f(x_2,z_2),f(y_2,z_2)),z_1) & & f(g(x_2,y_2),f(z_2,z_1)) \end{array}$$

The c.p. is convergent, as we have:

[1]The overlapping of a rule on itself in position ϵ is not to be considered.

$f(g(f(x_2, z_2), f(y_2, z_2)), z_1) \to_{r2,p=\epsilon} g(f(f(x_2, z_2), z_1), f(f(y_2, z_2), z_1))$
$\stackrel{+}{\to}_{r1,p=1,2} g(f(x_2, f(z_2, z_1)), f(y_2, f(z_2, z_1)))$
$f(g(x_2, y_2), f(z_2, z_1)) \to_{r2,p=\epsilon} g(f(x_2, f(z_2, z_1)), f(y_2, f(z_2, z_1)))$

When considering the other rules, we have the following critical pairs.

3. $cc(r3, r2)$ $p=1$ $\sigma = \{y_3/x_2,\ y_3/y_2\}$

$$\begin{array}{ccc} & f(g(y_3, y_3), z_2) & \\ \swarrow & & \searrow \\ f(y_3, z_2) & & g(f(y_3, z_2), f(y_3, z_2)) \end{array}$$

The c.p. is convergent, as we have:

$$g(f(y_3, z_2), f(y_3, z_2)) \to_{r3,p=\epsilon} f(y_3, z_2)$$

4. $cc(r4, r1)$ $p=1$ $\sigma = \{a/x_1,\ x_4/y_1\}$

$$\begin{array}{ccc} & f(f(a, x_4), z_1) & \\ \swarrow & & \searrow \\ f(x_4, z_1) & & f(a, f(x_4, z_1)) \end{array}$$

The c.p. is convergent, as we have:

$$f(a, f(x_4, z_1)) \to_{r4,p=\epsilon} f(x_4, z_1)$$

There are no further critical pairs. All critical pairs of R are convergent. Thus, by the Huet Lemma R is locally confluent.

Remark Exercise C13 in [1] is similar to exercise C3 above, as some rules are the same as in the TRS R in C3 except for the *order of the arguments*. This can lead to more or less cases of unification and critical pairs.

References

[1] M. Nesi, Esercizi di Riscrittura,
in `http://www.di.univaq.it/monica/MFI/EserciziR.pdf`.